Bulletin

of the

Calcutta Mathematical Society

Vol. VIII

(Nos. 1 and 2)

1916-17

CALCUTTA UNIVERSITY PRESS

PRINTED BY ATULCHANDRA BHATTACHARYYA
AT THE CALCUTTA UNIVERSITY PRESS, SENATE HOUSE, CALCUTTA

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Primitive matrices and the primitive degrees of a matrix.

Part II.

BY

C. E. CULLIS.

8. Invariance of the primitive degrees of a matrix in equigradent transformations.

The two theorems which follow will lead up to the general property of invariance contained in Theorem III.

Theorem I. *If $A=[a]_p^r$ and $B=[b]_m^r$ are two rational integral functional matrices, and if there exists an equigradent transformation of the form*

$$[h]_m^p\,[a]_p^r=[b]_m^r \tag{A}$$

where $[h]_m^p$ is an undegenerate matrix of rank p with constant elements, then:

(1) The degrees of the successive vertical rows are the same in both the matrices A and B.

(2) Each of the matrices A and B is vertically primitive when and only when the other is vertically primitive.

(3) The equigradent transformation which converts A into B also converts every vertically primitive matrix vertically equivalent to A into a vertically primitive matrix vertically equivalent to B.

Theorem II. *If $A=[a]_r^q$ and $B=[b]_r^n$ are two rational integral functional matrices, and if there exists an equigradent transformation of the form*

$$[a]_r^q\,[k]_q^n=[b]_r^n, \tag{B}$$

where $[k]_q^n$ is an undegenerate matrix of rank q with constant elements, then:

(1) The degrees of the successive horizontal rows are the same in both the matrices A and B.

(2) *Each of the matrices A and B is horizontally primitive when and only when the other is horizontally primitive.*

(3) *The equigradent transformation which converts A into B also converts every horizontally primitive matrix horizontally equivalent to A into a horizontally primitive matrix horizontally equivalent to B.*

We prove Theorem II only, the proof of Theorem I being similar. Let $\overset{q}{\underset{n}{\mathrm{K}}}$ be an inverse postfactor of the matrix $[k]_q^n$ in (B), it being a matrix of rank q with constant elements. Then from the equations

$$[b_{i1}\ b_{i2}\ \ b_{in}]=[a_{i1}\ a_{i2}\ \ a_{iq}]\ [k]_q^n$$

$$[a_{i1}\ a_{i2}\ \ a_{iq}]=[b_{i1}\ b_{i2}\ \ b_{in}]\ \overset{q}{\underset{n}{\mathrm{K}}}$$

we see that the degree of the i th horizontal row of B cannot exceed the degree of the i th horizontal row of A, and that the degree of the i th horizontal row of A cannot exceed the degree of the i th horizontal row of B. Therefore the i th horizontal rows of A and B have equal degrees.

Now suppose, as is clearly allowable, that the successive horizontal rows of A and B are so arranged that their degrees are in ascending order of magnitude. Then if

$$[v]_n=[u]_q\ [k]_q^n \qquad \text{so that} \qquad [u]_q=[v]_n\ \overset{q}{\underset{n}{\mathrm{K}}},$$

and if $\omega \neq 0$, each of the rational integral identities

$$[\phi_1\ \phi_2 \dots \phi_s]\ [a]_s^q=\omega\ [u_1\ u_2 \dots u_q],$$

$$[\phi_1\ \phi_2 \dots \phi_s]\ [b]_s^n=\omega\ [v_1\ v_2 \dots v_n]$$

is true when and only when the other is true. Since $[u]_q$ and $[v]_n$ have equal degrees, it follows from the test (C) in Art. 5 that B is horizontally primitive when and only when A is horizontally primitive.

Lastly, discarding the assumption that the degrees of the successive horizontal rows of A and B are in ascending order of magnitude, let ρ be the common rank of A and B, let $[a]_\rho^q$ be a horizontally primitive

matrix horizontally equivalent to A, and let $[\beta]_\rho^n$ be defined by the equation

$$[a]_\rho^q\ [k]_q^n = [\beta]_\rho^n .$$

Then there exist rational integral identities in the variables of the forms

$$\omega\, [a]_r^q = [l]_r^\rho\ [a]_\rho^q\ , \qquad \omega\, [b]_r^n = [l]_r^\rho\ [\beta]_\rho^n$$

where $[l]_r^\rho$ has rank ρ and $\omega \neq 0$, the second equation being a consequence of the first. The second equation shows that $[\beta]_\rho^n$ is horizontally equivalent to B, and by the second part of the theorem it is a horizontally primitive matrix.

Thus Theorem II is completely proved ; and we see also that A and B have the same horizontal primitive degrees.

Ex. i. If any matrix A is converted into a matrix B by derangements of horizontal rows, then the same derangements convert any vertically primitive matrix α vertically equivalent to A into a vertically primitive matrix β vertically equivalent to B.

This follows from Theorem I as a particular case. Clearly the degrees of the successive vertical rows are the same in both the matrices α and β.

Ex. ii. If any matrix A is converted into a matrix B by derangements of vertical rows, then the same derangements convert any horizontally primitive matrix α hori ontally equivalent to A into a horizontally primitive matrix β horizontally equivalent to B.

This follows from Theorem II as a particular case. Clearly the degrees of the successive horizontal rows are the same in both the matrices α and β.

Theorem III. *If $A=[a]_p^q$ and $B=[b]_m^n$ are two rational integral functional matrices, and if there e ists an equigradent transformation of the form*

$$[h]_m^p\ [a]_p^q\ [k]_q^n = [b]_m^n , \qquad \text{(C)}$$

where $[h]_m^p$ and $[k]_q^n$ are undegenerate matrices of ranks p and q with constant elements, then A and B have the same rank, the same potent divisors, and the same horizontal and vertical primitive degrees.

Since (C) is an equipotent transformation, A and B must be equipotent, *i. e.* they must have the same rank and the same potent divisors. It only remains to prove the last part of the theorem.

Let the common rank of A and B be r, and let $[a]_p^r$ be a vertically primitive matrix vertically equivalent to A. Then if

$$[h]_m^p\ [a]_p^r = [\beta]_m^r, \qquad [l\,]_r^q\,[k]_q^n = [\lambda]_r^n,$$

there exist rational integral identities in the variables of the forms

$$\omega\,[a]_p^q = [a]_p^r\ [l]_r^q \qquad \omega\ [b]_m^n = [\beta]_m^r\,[\lambda]_r^n,$$

where $[l]_r^q$ and $[\lambda]_r^n$ have rank r and where $\omega \neq 0$, the second eqation being a consequence of the first. The second identity shows that $[\beta]_m^r$ is vertically equivalent to B, and from Theorem II it follows that $[\beta]_m^r$ is a vertically primitive matrix whose successive vertical rows have the same degrees as the successive vertical rows of of $[a]_p^r$ Accordingly the vertical primitive degrees of A and B, being respectively those of $[a]_p^r$ and $[\beta]_m^r$, must be the same.

Again let $[a]_r^q$ be a horizontally primitive matrix horizontally equivalent to A, Then if

$$[a]_r^q\ [k]_q^n = [\beta]_r^n, \qquad [h]_m^p\,[l]_p^r = [\lambda]_m^r,$$

there exist rational integral identities in the variables of the forms

$$\omega\ [a]_p^q = [l]_p^r\ [a]_r^q, \qquad \omega\ [b]_m^n = [\lambda]_m^r\,[\beta]_r^n,$$

where $[l]_p^r$ and $[\lambda]_m^r$ have rank r and where $\omega \neq 0$, the second equation being a consequence of the first. The second identity shows that $[\beta]_r^n$ is horizontally equivalent to B, and from Theorem II it follows that $[\beta]_r^n$ is a horizontally primitive matrix whose successive horizontal rows have the same degrees as the successive horizontal rows

of $[a]_r^q$. Accordingly the horizontal primitive degrees of A and B, being respectively those of $[a]_r^q$ and $[\beta]_r^n$, must be the same.

Corollary. *If $A = [a]_p^q$ and $B = [b]_m^n$ are rational integral functional matrices, and if there exists a transformation of the form*

$$[h]_u^p \ [a]_p^q \ [k]_q^v \ = \ [h']_u^m \ [b]_m^n \ [k']_n^v, \qquad \text{(D)}$$

where the prefactors and postfactors on both sides are undegenerate matrices with constant elements whose ranks are equal to their passivities, then A and B have the same rank, the same potent divisors, and the same hori ontal primitive degrees.

For if $[c]_u^v$ is the common value of both-sides in the equation (D), both the matrices A and B can be converted into the same rational integral functional matrix $C = [c]_u^v$ by equigradent transformations, and they both have the same rank, the same potent divisors, and the same horizontal and vertical primitive degrees as C.

Ex. iii. The hori.ontal and vertical primitive degrees of any matrix remain invariant in all derangements of the matrix.

This is obvious, nnd is a particular case of Theorem III. Moreover we see from the proof of Theorem III that :

If any matrix A is converted into B by certain derangements h of its horizontal rows and certain derangements k of its vertical rows, then the derangements h convert every vertically primitive matrix a vertically equivalent to A into a vertically primitive matrix β vertically equivalent to B, and the derangements k convert every horizontally primitive matrix a horizontally equivalent to A into a horizontally primitive matrix β horizontally equivalent to B.

In each case if a is underanged, then β is underanged.

9. Primitive degrees and equivalent primitive matrices of a compartite matrix.

A compartite matrix is one in which all elements are zero except those lying in a number of mutually complementary minors, and these mutually complementary minors are called the parts of the matrix. In this article Φ will always denote a compartite matrix whose elements are rational integral functions of $x, y, z \ldots$.

We have in the first place the following two theorems :

The compartite matrix Φ is vertically primitive when and only when its parts are all vertically primitive. (A)

The compartite matrix Φ is hori ontally primitive when and only when its parts all horizontally primitive. (B)

It will be sufficient to prove the second of these theorems, the proof of the first theorem being similar.

If Φ is horizontally primitive, it follows from Ex. i of Art. 5 that its parts are all horizontally primitive. It remains to show that if the parts of Φ are all horizontally primitive, then Φ itself is horizontally primitive. We will suppose that Φ has three parts ; and by Ex. iii of Art. 8 there will be no loss of generality in supposing that it has the *standard form*

$$\Phi = \begin{bmatrix} a, & 0, & 0 \\ 0, & b, & 0 \\ 0, & 0, & c \end{bmatrix}_{\lambda,\ \mu,\ \nu}^{l,\ m,\ n} \tag{1}$$

Accordingly we will show that if the parts $A = [a]_\lambda^l$, $B = [b]_\mu^m$, $C = [c]_\nu^n$ of the compartite matrix Φ given by (1) are all horizontally primitive, then Φ itself is horizontally primitive. A similar method of proof is applicable whatever the number of parts may be,

Let $a_1, a_2, \dots a_\lambda ; \beta_1, \beta_2, \dots \beta_\mu ; \gamma_1, \gamma_2, \dots \gamma_\nu$ be the degrees of the successive horizontal rows of A, B, C ; and consider all rational integral identities in $x, y, z, \dots$ of the form

$$\begin{aligned} &[\theta_1\ \theta_2\ \dots\ \theta_\lambda\ \phi_1\ \phi_2\ \dots\ \phi_\mu\ \psi_1\ \psi_2\ \dots\ \psi_\nu].\Phi \\ &= [u_1\ u_2\ \dots\ u_l\ v_1\ v_2\ \dots\ v_m\ w_1\ w_2\ \dots\ w_n] \\ &= \omega\, \mathrm{H}, \end{aligned} \tag{2}$$

in which ω does not vanish identically. If Φ is not horizontally primitive, then by the test (B′) in Art. 5 one of the following three cases must occur :

CASE. I There is a relation of the form (2) in which θ_i does not vanish identically and H has degree less that a_i, i being some one of the integers 1, 2,...λ. In this case there must exist an identity of the form

$$[\theta_1\ \theta_2 \dots \theta_\lambda]\ [a]_\lambda^l = \omega\, [u_1\ u_2 \dots u_l]$$

in which θ_i does not vanish identically and $[u]_l$ has degree less than α_i; and the test (B′) of Art. 5 shows that A is not horizontally primitive.

CASE II. There is a relation of the form (2) in which ϕ_j does not vanish identically and H has degree less than β_j, j being some one of the integers $1, 2, \ldots \mu$. In this case there must exist an identity of the form

$$[\phi_1\ \phi_2 \ldots \phi_\mu]\ [b]_\mu^m = \omega\ [v_1\ v_2 \ldots v_m]$$

in which ϕ_j does not vanish identically and $[v]_m$ has degree less than β_j; and the test (B′) of Art, 5 shows that B is not horizontally primitive.

CASE III. There is a relation of the form (2) in which ψ_k does not vanish identically and H has degree less than γ_k, k being some one of the integers $1, 2, \ldots \nu$. In this case there must exist an identity of the form

$$[\psi_1\ \psi_2 \ldots \psi_\nu]\ [c]_\nu^n = \omega\ [w_1\ w_2 \ldots w_n]$$

in which ψ_k does not vanish identically and $[w]_n$ has degree less than γ_k; and the test (B′) of Art. 5 shows that C is not horizontally primitive.

Since A, B, C are all horizontally primitive, none of these cases are possible, and therefore Φ must be horizontally primitive.

We have next the two following theorems:

If Φ is a compartite matrix of standard form, and if ϕ is another compartite matrix of standard form whose successive parts are vertically primitive matrices vertically equivalent to the successive parts of Φ, then ϕ is a vertically primitive matrix vertically equivalent to Φ.

If Φ is a compartite matrix of standard form, and if ϕ is another compartite matrix of standard form whose successive parts are horizontally primitive matrices horizontally equivalent to the successive parts of Φ, then ϕ is a horizontally primitive matrix horizontally equivalent to Φ.

It will be sufficient to prove the second of these two theorems, the proof of the first theorem being similar.

Let Φ be again the matrix (1), its horizontal rows being now not necessarily unconnected; let the ranks of the parts A, B, C be respectively p, q, r; and let $[\alpha]_p^l$, $[\beta]_q^m$, $[\gamma]_r^n$ be horizontally primitive matrices horizontally equivalent to A, B, C respectively. Then there exist rational integral identities in x, y, z, of the forms

$$\omega\ [a]_\lambda^l = [u]_\lambda^p\ [\alpha]_p^l, \quad \omega\ [b]_\mu^m = [v]_\mu^q\ [\beta]_q^m, \quad \omega\ [c]_\nu^n = [w]_\nu^r\ [\gamma]_r^n,$$

$$\omega\,\Phi = \begin{bmatrix} u, & 0, & 0 \\ 0, & v, & 0 \\ 0, & 0, & w \end{bmatrix}^{p,\,q,\,r}_{\lambda,\,\mu,\,\nu} \begin{bmatrix} \alpha, & 0, & 0 \\ 0, & \beta, & 0 \\ 0, & 0, & \gamma \end{bmatrix}^{l,\,m,\,n}_{p,\,q,\,r},$$

in which $[u]^{p}_{\lambda}, [v]^{q}_{\mu}, [w]^{r}_{\nu}$ have ranks p, q, r, and $\omega \neq 0$, the last equation being a consequence of the three preceding equations. Since the prefactor on the right in the last equation in undegenerate and has rank $p+q+r$, that equation shows that the compartite matrix

$$\phi = \begin{bmatrix} \alpha, & 0, & 0 \\ 0, & \beta, & 0 \\ 0, & 0, & \gamma \end{bmatrix}^{l,\,m,\,n}_{p,\,q,\,r}$$

is horizontally equivalent to Φ, and by the second of the two theorems given at the commencement of this article it is a horizontally primitive matrix.

Thus the second of the above two theorems is true when Φ has three parts, and a similar proof is applicable whatever number of parts Φ may have.

Finally we have the following general theorem:

If Φ is any compartite matrix, not necessarily of standard form, then:

(1). *The rank of Φ is equal to the sum of ranks of its parts.*

(2). *The potent divisors of Φ are the potent divisors of its several parts.*

(3). *The horizontal and vertical primitive degrees of Φ are respectively the horizontal and vertical primitive degrees of its several parts.* (E)

We here assume that the first two parts of this theorem are known to be true, and prove only the last part of the theorem.

If Φ is of standard form, then in the theorem (C) the vertical primitive degrees of Φ are the vertical primitive degrees of ϕ, which are the vertical primitive degrees of the several parts of Φ; and in the theorem (D) the horizontal primitive degrees of Φ are the horizontal primitive degrees of ϕ, which are the horizontal primitive degrees of the several parts of Φ. Thus the theorem (E) is true when Φ is of standard form.

Now by Ex. iii of Art 8 the horizontal and vertical primitive degrees of Φ and its several parts remain unaltered in all derangements of Φ. Therefore the theorem (E) must also be true when Φ is not of standard form.

10. Primitive matrices equivalent to a matrix whose horizontal or vertical rows are all homogeneous.

We consider in this article the horizontally primitive matrices horizontally equivalent to a given matrix in which every horizontal row is homogeneous in the variables, and the vertically primitive matrices vertically equivalent to a given matrix in which every vertical row is homogeneous in the variables. One property of such matrices has already been given in Ex. ii of Art. 5.

The following preliminary examples relate to a matrix $A=[a]_r^n$ of rank r, rational and integral in the variables $x, y, z, \ldots$, in which each horizontal row is homogeneous in the variables. There are clearly corresponding results for a matrix whose vertical rows are unconnected and all homogeneous.

Ex. i. Every primitive one-rowed matrix of any degree ϵ connected with the horizontal rows of A is the sum of a number of homogeneous one-rowed matrices connected with the horizontal rows of A.

Let $[b]_n=[b, b_2 \ldots b_n]$ be a primitive one-rowed matrix of degree ϵ connected with the horizontal rows of A. Then since $[b]_n$ does not vanish identically and is impotent, there exists a rational integral identity of the form

$$\omega\,[b_1, b_2 \ldots b_n]=[f]_1^r\;[a]_r^n$$

$$=f_{11}\,[a_{11}\,a_{12}\ldots a_{1n}]+f_{12}\,[a_{21}\,a_{22}\ldots a_{2n}]+\ldots+f_{1r}\,[a_{r1}\,a_{r2}\ldots a_{rn}]; \quad (1)$$

where $[f]_1^r$ is impotent and $\omega\neq 0$. As in Art. 3 the function ω is a factor of E_r, the potent factor of A of order r, and since E_r is in this case homogeneous, ω must be homogeneous. Expressing $[b]_n$ in the form

$$[b]_n=[\beta]_n+[\beta']_n+[\beta'']_n+\cdots\;, \quad (2)$$

where the matrices on the right are homogeneous of successively diminishing degrees, and where $[\beta]_n$ is a homogeneous non-zero matrix of degree ϵ, we see by equating terms of the same degrees on both sides of (1) that there exist identities of the forms

$$\omega[\beta]_n=[\phi]_1^r\;[a]_r^n,\quad \omega\,[\beta']_n=[\phi']_1^r\;[a]_r^n,\quad \omega\,[\beta'']_n=[\phi'']_1^r\;[a]_r^n,\ldots .$$

Therefore $[\beta]_n$, $[\beta']_n$, $[\beta'']_n,\ldots$ are homogeneous one-rowed matrices connected with the horizontal rows of A, and we see from (2) that the theorem is true.

Ex. ii. If there is a primitive one-rowed matrix of any degree ϵ connected with the horizontal rows of A, *then there is a homogeneous non-zero one-rowed matrix of degree ϵ connected with the horizontal rows of* A.

For $[\beta]_n$ in Ex. i is a homogeneous non-zero one-rowed matrix connected with the horizontal rows of A.

Ex. iii. If there is any non-zero one-rowed matrix of degree ϵ connected with the horizontal rows of A, *then there is a homogeneous non-zero one-rowed matrix of degree ϵ connected with the horizontal rows of* A.

For if $[c]_n$ is a non-zero one-rowed matrix of any degree ϵ connected with the horizontal rows of A, and if $[b]_n$ is an equivalent primitive matrix having degree ϵ', where $\epsilon' \not> \epsilon$, then by Ex. ii we can determine a homogeneous non-zero one-rowed matrix of degree ϵ' connected with the horizontal rows of A, and when we multiply this by any non-vanishing homogeneous function of degree $\epsilon-\epsilon'$, we obtain a homogeneous non-zero one-rowed matrix of degree ϵ connected with the horizontal rows of A.

Ex. iv. Every non-zero one-rowed matri of the lowest possible degree connected with the horizontal rows of A *is homogeneous*

In Ex. i let ϵ be the lowest possible degree of a non-zero one-rowed matrix connected with the horizontal rows of A. Then the matrices $[\beta']_n, [\beta'']_n, \ldots$, whose degrees are less than ϵ, must all vanish, and we have $[b]_n = [\beta]_n$, *i.e.* $[b]_n$ is homogeneous.

From these examples we see that the determination of all one-rowed matrices connected with the horizontal rows of A can be reduced to the determination of all homogeneous primitive one-rowed matrices connected with the horizontal rows of A. Further from Ex. iii we see that in applying to the matrix A the tests for primitivity given in Art. 5 we can restrict $[h]_n$ and ϕ_0 and therefore $\phi_1, \phi_2, \ldots \phi_r$ to be homogeneous; in fact we can restrict ϕ_0 to be E_r and $[h]_n$ to be homogeneous and primitive.

We will now show that the following two theorems are true:

Theorem I. *If* $A=[a]_r^n$ *is a rational integral functional matrix whose horizontal rows are unconnected, and if each horizontal row is homogeneous in the variables, then we can determine a horizontally equivalent underanged primitive matrix in which each horizontal row is homogeneous in the variables.*

Theorem II. *If* $A=[a]_m^r$ *is a rational integral functional matrix whose vertical rows are unconnected, and if each vertical row is homogeneous in the variables, then we can determine a vertically equivalent underanged primitive matrix in which each vertical row is homogeneous in the variables.*

It will be sufficient to prove Theorem II only, the proof of Theorem I being similar.

Let the r vertical primitive degrees of A in Theorem II be

$$u \text{ equal to } \eta_1,\ v \text{ equal to } \eta_2,\ w \text{ equal to } \eta_3,\ \ldots\ ,$$

where $\eta_1, \eta_2, \eta_3, \ldots$ are unequal positive integers arranged in ascending order of magnitude, and where $u+v+w+\ldots=r$; and let

$$[b]^r_m=[b, c, d, \ldots]^{u,\,v,\,w,\,\ldots}_m\ , \qquad [\beta]^r_m=[\beta, \gamma, \delta, \ldots]^{u,\,v,\,w,\,\ldots}_m\ ,$$

where $[b]^r_m$ is an underanged vertically primitive matrix vertically equivalent to A, and where $[\beta]^u_m$, $[\gamma]^v_m$, $[\delta]^w_m$, ... are the homogeneous matrices of degrees $\eta_1, \eta_2, \eta_3, \ldots$ formed by the terms of highest degrees in $[b]^u_m$, $[c]^v_m$, $[d]^w_m$, ... We will show that $[b]^r_m$ and $[\beta]^r_m$ are mutually equivalent, and this will establish the theorem; for it then follows that $[\beta]^r_m$ is an underanged vertically primitive matrix vertically equivalent to A.

In the first place we have $[b]^u_m=[\beta]^u_m$, for by Ex. iv the matrix $[b]^u_m$ is homogeneous.

Now let E be the potent factor of A of order r, which in the present case is necessarily homogeneous, and let $[c]^v_m=[\gamma]^v_m+[\gamma']^v_m$, so that $[\gamma']^v_m$ has degree less than η_2. By Art. 3 there exists a rational integral identity in the variables of the form

$$\mathrm{E}\,[c]^v_m=[a]^r_m\ [f]^v_r\ , \tag{3}$$

and by equating the terms of highest degrees in every pair of corresponding vertical rows on both sides of (3) we see that there exist identities of the forms

$$\mathrm{E}\,[\gamma]^v_m=[a]^r_m\ \ [\phi]^v_r\ , \qquad \mathrm{E}\,[\gamma']^v_m=[a]^r_m\ \ [\phi']^v_r\ .$$

The second of these equations shows that all the vertical rows of $[\gamma']^v_m$, *each of which has degree less than* η_2, are connected with the vertical rows of A, and therefore with the vertical rows of $[b]^u_m$ or $[\beta]^u_m$. Therefore there exists an identity of the form

$$\omega[\gamma']^v_m=[\beta]^u_m\,[\lambda]^v_u\ ,$$

where $\omega \neq 0$; and we then have

$$\omega\,[b,\,c]_m^{u,\,v} = \omega\,[\beta,\,c]_m^{u,\,v} = [\beta,\,\gamma]_m^{u,\,v} \begin{bmatrix} \omega, \lambda \\ O, \omega \end{bmatrix}_{u,\,v}^{u,\,v}.$$

Consequently $[b,\,c]_m^{u,\,v}$ and $[\beta,\,\gamma]_m^{u,\,v}$ are mutually equivalent.

Next let $[d]_m^w = [\delta]_m^w + [\delta']_m^w$, so that $[\delta']_m^w$ has degree less than η_3.

By Art. 3 there exists a rational integral identity in the variables of the form

$$\mathrm{E}\,[d]_m^w = [a]_m^r\ [f]_r^w,$$

from which, as in the preceding step of the proof, we deduce identities of the forms

$$\mathrm{E}\,[\delta]_m^w = [a]_m^r\ [\phi]_r^w, \qquad \mathrm{E}\,[\delta']_m^w = [a]_m^r\ [\phi']_r^w.$$

The second of these equations shows that all the vertical rows of $[\delta']_m^w$, *each of which has degree less than* η_3, are connected with the vertical rows of A, and therefore with the vertical rows of $[b,\,c]_m^{u,\,v}$ and $[\beta,\gamma]_m^{u,\,v}$. Therefore there exists an identity of the form

$$\omega\,[\delta']_m^w = [\beta,\,\gamma]_m^{u,\,v} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{u,\,v}^{w},$$

where $\omega \neq 0$, and we then have

$$\omega\,[\beta,\,\gamma,\,d]_m^{u,\,v,\,w} = [\beta,\,\gamma,\,\delta]_m^{u,\,v,\,w} \begin{bmatrix} \omega, O, \lambda \\ O, \omega, \mu \\ O, O, \omega \end{bmatrix}_{u,\,v,\,w}^{u,\,v,\,w}.$$

Consequently $[\beta,\,\gamma,\,d]_m^{u,\,v,\,w}$ and $[\beta,\,\gamma,\,\delta]_m^{u,\,v,\,w}$, and therefore also $[b,\,c,\,d]_m^{u,\,v,\,w}$ and $[\beta,\,\gamma,\,\delta]_m^{u,\,v,\,w}$, are mutually equivalent.

Continuing in this way, we see finally that $[b]_m^r$ and $[\beta]_m^r$ are mutually equivalent, which proves the theorem.

Corollary. *If A lies in a restricted domain of rationality Ω, then in Theorem I we can determine a horizontally equivalent underanged primitive matrix lying in Ω in which each horizontal row is homogeneous in the variables, and in Theorem II we can determine a vertically equivalent underanged primitive matrix lying in Ω in which each vertical row is homogeneous in the variables.*

For from Art. 7 we see that in the proof of Theorem II given above the matrix $[b]_m^r$ can be so chosen as to lie in Ω.

All the results of this article which are true for a matrix $A=[a]_r^n$ of rank r whose horizontal rows are homogeneous are clearly also true for a matrix $A=[a]_m^n$ of rank r whose horizontal rows are homogeneous, for in proving them for the latter matrix we can replace that matrix by any undegenerate horizontal minor of rank r. Similarly all the results which are true for a matrix $A=[a]_m^r$ of rank r whose vertical rows are homogeneous are also true for a matrix $A=[a]_m^n$ of rank r whose vertical rows are homogeneous.

Ex. v. Let $A=[a]_m^n$ be a matrix of rank r in which every vertical row is homogeneous in the variables, and let $B=[b]_m^r$ be any vertically equivalent vertically primitive matrix. Then the matrix $[\beta]_m^r$ formed by retaining only the terms of highest degree in every vertical row of B is undegenerate and primitive, and is vertically equivalent to A.

This follows from the proof of Theorem II.

Ex. vi. Let $A=[a]_m^n$ be a matrix of rank r in which every horizontal row is homogeneous in the variables, and let $B=[b]_r^n$ be any horizontally equivalent horizontally primitive matrix. Then the matrix $[\beta]_r^n$ formed by retaining only the terms of highest degree in every vertical row of B is undegenerate and primitive, and is horizontally equivalent to A.

11. Invariance of the primitive degrees of a matrix in homogeneous linear transformations of the variables.

The equations of an *ordinary* homogeneous linear transformation are of the forms

$$\overline{\underline{x}}_p = [l]_p^p\,[y]_p, \qquad \overline{\underline{y}}_p = \overline{\underline{L}}_p^p\,\overline{\underline{x}}_p, \tag{A}$$

where $[l]_p^p$ and $\overline{\underline{L}}_p^p$ are two mutually inverse undegenerate square matrices with constant elements, each equation being true when and only when the other is true. We use the first equation to express functions of $x_1, x_2, \dots x_p$ as functions of $y_1, y_2, \dots y_p$, and the second

equation to express functions of $y_1, y_2, \ldots y_p$ as functions of $x_1, x_2, \ldots x_p$. The two substitutions (A) establish a one-one correspondence between

(1) all rational integral functions ϕ of the p variables $x_1, x_2, \ldots x_p$,

(2) all rational integral functions ψ of the p variables $y_1, y_2, \ldots y_p$;

and they also establish a one-one correspondence between

(1′) all matrices A whose elements are rational integral functions of the p variables $x_1, x_2, \ldots x_p$,

(2′) all matrices B whose elements are rational integral functions of the p variables $y_1, y_2 \ldots y_p$;

two such functions or two such matrices corresponding when and only when they are convertible into one another by the substitutions (A). Two corresponding functions ϕ and ψ have equal degrees in all their variables, and each of them vanishes identically when and only when the other vanishes identically. Two corresponding matrices A and B (which are necessarily similar, *i.e.* have the same orders) have equal degrees and equal ranks; moreover there is a one-one correspondence between all the potent divisors of A and all the potent divisors of B, two corresponding potent divisors of A and B being convertible into one another by the substitutions (A).

In the following (a_{ij}, b_{ij}), (h_i, k_i), (ϕ_i, ψ_i), (ϕ_{ij}, ψ_{ij}), $(\alpha_{ij}, \beta_{ij})$ will always denote pairs of corresponding functions convertible into one another by the substitutions (A); the first function in each pair being rational and integral in $x_1, x_2, \ldots x_p$; the second function in each pair being rational and integral in $y_1, y_2, \ldots y_p$; and the two functions in each pair having equal degrees, and each of them vanishing identically when and only when the other vanishes identically.

Now let $A=[a]_m^n$ and $B=[b]_m^n$ be any two matrices which correspond according to the above definition; the elements of A being rational integral functions of the x's; the elements of B being rational integral functions of the y's; and A and B being convertible into one another by the substitutions (A). These two matrices have equal ranks, and similarly situated horizontal and vertical rows in A and B have equal degrees. We will consider the relations existing between undegenerate matrices equivalent to A and undegenerate matrices equivalent to B.

First let $[h]_n$ and $[k]_n$ be two corresponding one-rowed matrices convertible into one another by the substitutions (A), and let $\phi_0 \neq 0$, so that $\psi_0 \neq 0$. Then identities of the forms

$$\phi_0 [h_1\ h_2 \ldots h_n] = [\phi_1\ \phi_2 \ldots \phi_m]\ [a]_m^n$$

$$\psi_0 [k_1, k_2, \ldots k_n] = [\psi_1\ \psi_2 \ldots \psi_m]\ [b]_m^n \qquad (1)$$

occur in corresponding pairs. Since $[h]_n$ and $[k]_n$ have equal degrees, and $[k]_n$ is primitive when and only when $[h]_n$ is primitive, we conclude that:

There is a one-one correspondence between all one-rowed primitive matrices connected with the horizontal rows of A and all one-rowed primitive matrices connected with the horizontal rows of B, two such corresponding one-rowed primitive matrices having equal degrees and being convertible into one another by the substitutions (A). (B)

Because similarly situated horizontal rows of A and B have equal degrees, and because in (1) the function ψ_1 vanishes identically when and only when the function ϕ_1 vanishes identically, it follows further from the tests (B) and (B′) in Art. 5 that:

Each of the two corresponding matrices A and B is an underanged horizontally primitive matrix when and only when the other is an underanged horizontally primitive matrix. (C)

Next let the common rank of A and B be r, and let $[a]_r^n$ and $[\beta]_r^n$ be corresponding matrices, rational and integral in the x's and y's respectively, which are convertible into one another by the substitutions (A); and as before let $\phi_0 \neq 0$ so that $\psi_0 \neq 0$. Then identities of the forms

$$\phi_0\ [a]_m^n = [\phi]_m^r\ [\alpha]_r^n, \qquad \psi_0\ [b]_m^n = [\psi]_m^r\ [\beta]_r^n \qquad (2)$$

occur in corresponding pairs, each of them being true when and only when the other is true. Since $[\psi]_m^r$ has rank r when and only when $[\phi]_m^r$ has rank r, it follows from (2) that:

There is a one-one correspondence between all undegenerate matrices $[a]_r^n$ horizontally equivalent to A and all undegenerate matrices $[\beta]_r^n$ horizontally equivalent to B, any two such corresponding undegenerate matrices being convertible into one another by the substitutions (A), and similarly situated horizontal rows in them having equal degrees. (D)

Further since it has been shown in (C) that $[\beta]^n_r$ is an underanged horizontally primitive matrix when and only when $[a]^n_r$ is an underanged horizontally primitive matrix, it follows from (2) that:

There is a one-one correspondence between all underanged horizontally primitive matrices $[a]^n_r$ *horizontally equivalent to A and all underanged horizontally primitive matrices* $[\beta]^n_r$ *horizontally equivalent to B, two such corresponding primitive matrices being convertible into one another by the substitutions* (*A*), *and similarly situated horizontal rows in them having equal degrees.* (E)

The above properties remain true when we replace 'horizontal' and 'horizontally' by 'vertical' and 'vertically,' and $[a]^n_r$ and $[\beta]^n_r$ by $[a]^r_m$ and $[\beta]^r_m$. Hence from (D) and (E) we deduce that:

The two corresponding matrices A and B have the same horizontal and vertical primitive degrees. (F)

12. Invariance of the primitive degrees of a matrix when it is rendered homogeneous by a linear transformation of the variables.

We consider now *ordinary* linear transformations of the forms

$$\left\lceil\begin{matrix} x \\ 1 \end{matrix}\right\lfloor_{p,1} = [l]^{p+1}_{p+1} \left\lceil y \right\lfloor_{p+1}, \qquad \left\lceil y \right\lfloor_{p+1} = \left\lceil L \right\lfloor^{p+1}_{p+1} \left\lceil\begin{matrix} x \\ 1 \end{matrix}\right\lfloor_{p,1}, \tag{A}$$

where $[l]^{p+1}_{p+1}$ and $\left\lceil L \right\lfloor^{p+1}_{p+1}$ are two mutually inverse undegenerate square matrices with constant elements, each equation being true when and only when the other equation is true. We can use the first substitution to replace any rational integral function of $x_1, x_2, \ldots x_p$ by a homogeneous rational integral function of $y_1, y_2, \ldots y_{p+1}$, and the second substitution to replace any homogeneous rational integral function of $y_1, y_2, \ldots y_{p+1}$ by a rational integral function of $x_1, x_2, \ldots x_p$. These two substitutions establish a one-one correspondence between

(1) all rational integral functions ϕ of the p variables $x_1, x_2, \ldots x_p$,

(2) all those homogeneous rational integral functions ψ of the $p+1$ variables $y_1, y_2, \ldots y_{p+1}$ which are not divisible by the homogeneous linear function

$$Y = l_{p+1,1}\, y_1 + l_{p+1,2}\, y_2 + \ldots + l_{p+1,p+1}\, y_{p+1}\,; \tag{1}$$

and they also establish a one-one correspondence between

(1′) all matrices A whose elements are rational integral functions of the p variables $x_1, x_2, \ldots x_p$,

(2′) all homogeneous matrices B′ not divisible by Y whose elements are homogeneous rational integral functions of the $p+1$ variables $y_1, y_2, \ldots y_{p+1}$;

two such functions or two such matrices corresponding when and only when they are convertible into one another by the substitutions (A). Two corresponding functions ϕ and ψ have equal degrees in all their variables, and each of them vanishes identically when and only when the other vanishes identically. Two corresponding matrices A and B′, which are necessarily similar, have equal degrees and equal ranks; moreover there is a one-one correspondence between all the potent divisors of A and all those potent divisors of B′ which are not powers of Y, corresponding potent divisors of A and B′ being convertible into one another by the substitutions (A).

In the following (a_{ij}, b_{ij}), (h_i, k_i), (ϕ_i, ψ_i), (ϕ_{ij}, ψ_{ij}), $(\alpha_{ij}, \beta_{ij})$ will always denote pairs of corresponding functions convertible into one another by the substitutions (A); the first function in each pair being a rational integral function of $x_1, x_2, \ldots x_p$; the second function in each pair being a homogeneous rational integral function of $y_1, y_2, \ldots y_{p+1}$ which is not divisible by Y; and the two functions in each pair having equal degrees, and each vanishing identically when and only when the other vanishes identically. Further b'_{ij}, k'_i, ψ'_i, ψ'_{ij}, β'_{ij} will always be obtained by multiplying b_{ij}, k_i, ψ_i, ψ_{ij}, β_{ij} by positive integral powers of Y.

Now let $A = [a]_m^n$ and $B' = [b']_m^n$ be two similar matrices in which *similarly situated horizontal rows correspond* according to the above definition; the ith horizontal row of A being rational and integral in the x's; the i th horizontal row of B′ being rational, integral and homogeneous in the y's and not divisible by Y; and the i th horizontal rows of A and B′ having equal degrees and being convertible into one another by the substitutions (A). The matrices A and B′ have equal ranks, because B′ clearly has the same rank as the matrix formed when we render A completely homogeneous by means of the first of the substitutions (A). Every element of B′ is given by an equation of the form $b'_{ij} = Y^{\tau}\, b_{ij}$, where τ is a positive integer which has the value 0 for at least one element in every horizontal row of B′, and is the amount by which the degree of a_{ij} falls short of the degree of

$[a_{i1}\ a_{i2} \ldots a_{in}]$. We will consider the relations existing between undegenerate matrices horizontally equivalent to A and undegenerate matrices horizontally equivalent to B′.

First let $[h]_n$ and $[k']_n$ be two corresponding one-rowed matrices of equal degrees convertible into one another by the substitutions (A), $[k']_n$ being homogeneous in the y's and not divisible by Y; and let $\phi_0 \neq 0$ so that $\psi_0 \neq 0$. Then identities of the forms

$$\phi_0\ [h]_n = \phi_1\ [a_{11}\ a_{12} \ldots a_{1n}] + \phi_2\ [a_{21}\ a_{22} \ldots a_{2n}] + \ldots$$
$$+ \phi_m\ [a_{m1}\ a_{m2} \ldots a_{mn}] = [\phi_1\ \phi_2 \ldots \phi_m]\ [a]_m^n \qquad (2)$$

$$\psi_0\ [k']_n = \mathrm{Y}^{\tau_1} \psi_1\ [b'_{11}\ b'_{12} \ldots b'_{1n}] + \mathrm{Y}^{\tau_2} \psi_2\ [b'_{21}\ b'_{22} \ldots b'_{2n}]$$
$$+ \ldots + \mathrm{Y}^{\tau_m} [b'_{m1}\ b'_{m2} \ldots b'_{mn}] = [\psi'_1\ \psi'_2 \ldots \psi'_m]\ [b']_m^n \qquad (2')$$

occur in corresponding pairs; $\tau_1, \tau_2, \ldots \tau_m$ being positive integers, at least one of which is 0, so chosen that the matrix on the right in (2′) is homogeneous and of the same degree as the matrix on the left; so that τ_j is the amount by which the degree of $\psi_j\ [b'_{j1}\ b'_{j2} \ldots b'_{jn}]$ falls short of the degree of $\psi_0\ [k']_n$, which is also the amount by which the degree of $\phi_j\ [a_{j1}\ a_{j2} \ldots a_{jn}]$ falls short of the degree of $\phi_0\ [h]_n$. Since $[k']_n$ is primitive when and only when $[h]_n$ is primitive, we conclude that:

There is a one-one correspondence between all one-rowed primitive matrices connected with the horizontal rows of A and all homogeneous one-rowed primitive matrices connected with the horizontal rows of B′, two such corresponding one-rowed primitive matrices having equal degrees and being convertible into one another by the substitutions (A) when so applied as to cause no change in degree. (B)

Since similarly situated horizontal rows of A and B′ have equal degrees, and since in (2) and (2′) the function ψ'_j vanishes when and only when ϕ_j vanishes, and $[h]_n$ and $[k']_n$ have equal degrees, it follows from the tests (B) and (B′) in Art. 5 when we use Ex. 3 of Art. 10, that:

Each of the two matrices A and B′ is an underanged horizontally primitive matrix when and only when the other is an underanged horizontally primitive matrix. (C)

Next let the common rank of A and B′ be r, and let $[\alpha]_r^n$ and $[\beta']_r^n$ be two matrices whose similarly situated horizontal rows have equal

degrees and correspond in the same way as similarly situated horizontal rows of A and B′. We can choose $[a]_r^n$ to be an underanged horizontally primitive matrix horizontally equivalent to A; also, by Art. 10, we can as an alternative choose $[\beta']_r^n$ to be an underanged horizontally primitive matrix horizontally equivalent to B′. Then if $\phi_\circ \neq 0$, so that $\psi_\circ \neq 0$, we see from (2) and (2′) that identities of the forms

$$\phi_\circ [a]_m^n = [\phi]_m^r [a]_r^n, \qquad \psi_\circ [b']_m^n = [\psi']_m^r [\beta']_r^n \tag{3}$$

occur in corresponding pairs. Here we have $\psi' = Y^{\tau_{ij}} \psi_{ij}$, where the index τ_{ij} is such that $\psi'_{ij} [\beta_{j1}\, \beta_{j2} \ldots \beta_{jn}]$ has the same degree as $\psi_\circ [b'_{i1}\, b'_{i2} \ldots b'_{in}]$; and no horizontal row of $[\psi']_m^r$ is divisible by Y. Since $[\psi']_m^r$ has rank r when and only when $[\phi]_m^r$ has rank r, it follows from (3) that:

There is a one-one correspondence between all undegenerate matrices $[a]_r^n$ horizontally equivalent to A and all those undegenerate matrices $[\beta']_r^n$ horizontally equivalent to B′ in which every horizontal row is homogeneous and not divisible by Y, two such corresponding undegenerate matrices being convertible into one another by the substitutions (A) when so applied as not to change the degree of any horizontal row. (D)

Further since it has been shown in (C) that $[\beta']_r^n$ is an underanged horizontally primitive matrix when and only when $[a]_r^n$ is an underanged horizontally primitive matrix, it follows from (3) that:

There is a one-one correspondence between all underanged horizontally primitive matrices $[a]_r^n$ horizontally equivalent to A and all those underanged horizontally primitive matrices $[\beta']_r^n$ horizontally equivalent to B′ in which every horizontal row is homogeneous, two such corresponding primitive matrices being convertible into one another by the substitutions (A) when so applied as not to change the degree of any horizontal row. (E)

Consequently the two matrices A and B′ have the same horizontal primitive degrees. (F)

If B″ is any matrix formed by homogenising the horizontal rows of A by means of the first of the substitutions (A), the degree of each horizontal row being not necessarily unchanged, then every horizontal row of B″ is obtained by multiplying the similarly situated horizontal

row of B' by some power of Y, and B'' is therefore horizontally equivalent to B'. Hence all the results obtained above are still true when A and B' are any two similar matrices whose similarly situated horizontal rows are convertible into one another by the substitutions (A), provided only that every horizontal row of B' is homogeneous in the y's. In particular they are true when B' is completely homogeneous.

There are clearly corresponding results for two similar matrices A and B' whose similarly situated vertical rows are convertible into one another by the substitutions (A), the vertical rows of B' being all homogeneous in the y's.

In the particular case when A and B' are convertible into one another by the substitutions (A), and B' is completely homogeneous in the y's, the matrices A and B' have both the same horizontal and the same vertical primitive degrees. (G)

The chief utility of the results of this article lies in the fact that in determining the horizontal (or vertical) primitive degrees of a matrix we can always as a preparatory step render every horizontal (or vertical) row homogeneous in the variables.

NOTE. *Homogenisation by the introduction of a new variable.*

In the particular case when $[l]_{p+1}^{p+1} = [1]_{p+1}^{p+1}$, we can replace the two mutually inverse substitutions (A) by

$$1 = x_{p+1}, \qquad x_{p+1} = 1 \qquad \text{(A}'\text{)}$$

Then homogenisation of any function or matrix which is rational and integral in the p variables $x_1, x_2, \ldots x_p$ by the first of the substitutions (A′) is homogenisation by the introduction of the new variable x_{p+1}. In this case we must replace Y by x_{p+1}. The results (B), (C), (D), (E), (F), (G) then remain true when A and B' are two similar matrices whose similarly situated horizontal rows are convertible into one another by the substitutions (A′), the horizontal rows of B' being all homogeneous in the $p+1$ variables $x_1, x_2, \ldots x_{p+1}$; and there are corresponding results when A and B' are two similar matrices whose similarly situated vertical rows are convertible into one another by the substitutions (A′), the vertical rows of B' being all homogeneous in the $p+1$ variables $x_1, x_2 \ldots x_{p+1}$. In the particular case when A and B' are convertible into one another by the substitutions (A′), and B' is completely homogeneous in $x_1, x_2, \ldots x_{p+1}$, the matrices A and B' have both the same horizontal and the same vertical primitive degrees.

13. Invariance of the primitive degrees of a matrix in non-homogeneous linear transformations of the variables.

An *ordinary* non-homogeneous linear transformation of the p variables $x_1, x_2, \dots x_p$ is given by p equations of the form

$$
\begin{aligned}
x_1 &= l_{11}\, y_1 + l_{12}\, y_2 + \dots + l_{1p}\, y_p + u_{11}, \\
x_2 &= l_{21}\, y_1 + l_{22}\, y_2 + \dots + l_{2p}\, y_p + u_{21}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
x_p &= l_{p1}\, y_1 + l_{p2}\, y_2 + \dots + l_{pp}\, y_p + u_{p1},
\end{aligned}
$$

where $[l]_p^p$ is an undegenerate square matrix with constant elements, and where the u's are constants which are not all zero.

The matrix equations of this transformation and its inverse are

$$
\begin{bmatrix} x \\ 1 \end{bmatrix}_{p,\,1} = \begin{bmatrix} l, & u \\ 0, & 1 \end{bmatrix}_{p,\,1}^{p,\,1} \begin{bmatrix} y \\ 1 \end{bmatrix}_{p,\,1}, \qquad \begin{bmatrix} y \\ 1 \end{bmatrix}_{p,\,1} = \begin{bmatrix} \mathrm{L}, & \mathrm{U} \\ 0, & 1 \end{bmatrix}_{p,\,1}^{p,\,1} \begin{bmatrix} x \\ 1 \end{bmatrix}_{p,\,1}, \tag{A}
$$

where $[\mathrm{L}]_p^p$ is the inverse matrix of $[l]_p^p$, and $[\mathrm{U}]_p^1 = -[\mathrm{L}]_p^p\,[u]_p^1$, each of these equations being true when and only when the other is true. We can use the first of the substitutions (A) to express any rational integral function of the p variables $x_1, x_2, \dots x_p$ as a function of the p variables $y_1, y_2, \dots y_p$; and we can use the second substitution to express any rational integral function of $y_1, y_2, \dots y_p$ as a function of $x_1, x_2, \dots x_p$.

The properties of the transformations (A) can be deduced from the properties of the two mutually inverse homogeneous linear transformations

$$
[x]_{p+1} = \begin{bmatrix} l, & u \\ 0, & 1 \end{bmatrix}_{p,\,1}^{p,\,1} [y]_{p+1}, \qquad [y]_{p+1} = \begin{bmatrix} \mathrm{L}, & \mathrm{U} \\ 0, & 1 \end{bmatrix}_{p,\,1}^{p,\,1} [x]_{p+1}. \tag{A$'$}
$$

For if ϕ is any rational integral function of $x_1, x_2, \dots x_p$; ϕ' the function obtained when we homogenise ϕ without change of degree by the introduction of the new variable x_{p+1}; and if ψ is any rational integral function of $y_1, y_2 \dots y_p$; ψ' the function obtained when we homogenise ψ without change of degree by the introduction of the new variable y_{p+1}; then ϕ and ψ are convertible into one another by the substitutions (A) when and only when ϕ' and ψ' are convertible into one another by the substitutions (A$'$).

It follows therefore from the preceding two articles that the two substitutions (A) establish a one-one correspondence between

(1) all rational integral functions ϕ of the p variables $x_1, x_2, \dots x_p$,

(2) all rational integral functions ψ of the p variables $y_1, y_2, \dots y_p$;

and that they also establish a one-one correspondence between

(1′) all matrices A whose elements are rational integral functions of the p variables $x_1, x_2, \dots x_p$,

(2′) all matrices B whose elements are rational integral functions of the p variables $y_1, y_2, \dots y_p$;

two such functions ϕ and ψ or two such matrices A and B corresponding when and only when they are convertible into one another by the substitutions (A). Two corresponding functions ϕ and ψ or two corresponding matrices A and B have the same properties as in Art. 11.

In particular two corresponding matrices $A = [a]_m^n$ and $B = [b]_m^n$ convertible into one another by the substitutions (A) have the same horizontal and vertical primitive degrees.

It appears then that the primitive degrees of a matrix remain unaltered in all ordinary linear transformations of the variables of the types considered in this and the preceding two articles.

14. Relations between primitivity and impotence.

The relations for matrices whose elements are rational integral functions of a single variable will be deduced from the following auxiliary theorem:

If $A = [a]_r^n$ is an undegenerate matrix of rank r whose elements are rational integral functions of the single variable x, and if $[a]_{r-1}^n$ is impotent but not $[a]_r^n$, then there exists a rational integral identity in x of the form

$$[\phi_1\ \phi_2\ \dots\ \phi_{r-1}\ 1]\ [a]_r^n = \Delta\ [u_1\ u_2\ \cdots\ u_n], \tag{1}$$

where Δ is the H. C. F. of the simple minor determinants of A, $[\phi]_{r-1}$ has a lower degree in x than Δ, and $[u]_n$ is primitive and has a lower degree in x than $[a]_r^n$. (a)

Let $\nu = \binom{n}{r}$, and let $[A]_r^{\nu}$ be a complete matrix of the affected minor determinants of A of order $r-1$ in which the 1st, 2nd, ... r th horizontal rows are composed of the simple minor determinants of

the minor matrices of A formed by striking out its 1st, 2nd, ... r th horizontal rows. Then we have

$$\overset{\longrightarrow r}{\underset{\longrightarrow \nu}{A}} [a]_r^n = [b]_\nu^n = \Delta\, [\beta]_\nu^n \,, \tag{2}$$

where the elements of $[b]_\nu^n$ which do not vanish identically are the simple minor determinants of A, all of which are divisible by Δ. Since the elements of the last vertical row of $\overset{\longrightarrow r}{\underset{\longrightarrow \nu}{A}}$, which are the simple minor determinants of $[a]_{r-1}^n$, have no factor in common other than a non-vanishing constant, there exist rational integral functions $g_1, g_2, \ldots g_\nu$ such that

$$g_1\, A_{r1} + g_2\, A_{r2} + \ldots + g_\nu\, A_{r\nu} \doteq 1\,;$$

and when we prefix the matrix $[g]_\nu$ on both sides of (2), we obtain an identity of the form

$$[h, 1]_1^{r-1,\, 1}\ [a]_r^n = \Delta\, [c]_1^n \,. \tag{3}$$

Dividing every element of $[h]_1^{r-1}$ by Δ, we can write

$$[h]_1^{r-1} = \Delta\, [k]_1^{r-1} + [\phi]_1^{r-1} \,,$$

where $[\phi]_1^{r-1}$ has a lower degree in x than Δ; and when we substitute this value in (3), we obtain an identity of the form

$$[\phi, 1]_1^{r-1,\, 1}\ [a]_r^n = \Delta\, [u]_1^n \,,$$

i.e. we obtain an identity of the form (1) in which $[\phi]_{r-1}$ has a lower degree in x than Δ. It follows from Art. 3 that $[u]_n$ is then primitive. Moreover $[u]_n$ must have a lower degree in x than $[a]_r^n$, as otherwise the right-hand side of (1) would have a greater degree in x than the left-hand side. It can easily be shown that the identity (1) is unique, *i.e.* there cannot be two such identities.

In the excluded case when $[a]_r^n$ is impotent as well as $[a]_{r-1}^n$ we have $[\phi]_{r-1} = 0$ in (1).

Making use of the theorem (a) we can now prove that:

Every primitive matrix whose elements are rational integral functions of a single variable x must be impotent; but an undegenerate and impotent matrix of the same character is not necessarily primitive. (*A*)

Let $A = [a]_r^n$ be a horizontally primitive matrix of rank r whose elements are rational integral functions of the single variable x, and let its horizontal rows be so arranged that their degrees in x, starting from the first, are in ascending order of magnitude. Then the first horizontal row $[a]_1^n$ is necessarily undegenerate and impotent.

Now let i be any one of the integers $2, 3, \ldots r$, and let it be assumed that $[a]_{i-1}^n$ is impotent. If $[a]_i^n$ is not also impotent, let Δ be the H. C. F. of its simple minor determinants. Then by the auxiliary theorem (a) there exists a rational integral identity in x of the form

$$[\phi_1 \; \phi_2 \ldots \phi_{i-1} \; 1] \; [a]_i^n = \Delta \; [u_1 \; u_2 \ldots u_n]$$

in which $[u]_n$ has a lower degree in x than $[a]_i^n$, *i.e.* than the last horizontal row of $[a]_i^n$; but this is impossible because (see Ex. *i* of Art. 5) the matrix $[a]_i^n$ is primitive. Consequently if $[a]_{i-1}^n$ is impotent, then $[a]_i^n$ is impotent.

Since $[a]_1^n$ is impotent, it follows by putting $i = 2, 3, \ldots r$ in succession that $[a]_2^n, [a]_3^n, \ldots [a]_r^n$ are all impotent, *i.e.*, A is impotent.

Thus the first part of the theorem is true for horizontally primitive matrices, and can be shown in a similar way to be true for vertically primitive matrices.

To estabish the second part of the theorem it is sufficient to observe that the matrix

$$\begin{bmatrix} x, & 1, & 0 \\ x, & 0, & 1 \end{bmatrix} = \begin{bmatrix} 0, \\ -1, \end{bmatrix} \begin{bmatrix} 0, & 1, & -1 \\ x, & 1, & 0 \end{bmatrix},$$

which is undegenerate and impotent, is not primitive; for the post-factor on the right is an equivalent primitive matrix.

For convenience we will repeat here two theorems which have been established in Ex. *ii*. of Art. 5.

If $A = [a]_r^n$ *is an undegenerate matrix of rank r each of whose horizontal rows is homogeneous in the variables, which may be any in number, and if A is impotent, then it is also primitive.* (B)

If $A = [a]_m^r$ *is an undegenerate matrix of rank r each of whose vertical rows is homogeneous in the variables, which may be any in number, and if A is impotent, then it is also primitive.* (B′)

That the converses of these two theorems are not true in general will be clear from Ex. *i* below. But the converses are true when there are only two variables. In fact we can show that:

If there are only two variables x and y, and if $A = [a]_r^n$ *is an undegenerate matrix of rank r each of whose horizontal rows is homogeneous in x and y, then A is primitive when and only when it is impotent.* (C)

If there are only two variables x and y, and if $A = [a]_m^r$ *is an undegenerate matrix of rank r each of whose vertical rows is homogeneous in x and y, then A is primitive when and only when it is impotent.* (C′)

It will be sufficient to prove the first of these two theorems.

First let the matrix $A = [a]_r^n$ be impotent. Then from (B) we see that it is primitive.

Next let the matrix $A = [a]_r^n$ be primitive, and let it be denoted by $f(x, y)$. Then by Art. 12 the matrix $f(x, 1)$ is primitive, and it follows from (A) that $f(x, 1)$ is impotent; therefore by Art. 12 the matrix $f(x, y)$ can have no irresoluble divisor other than y; for if $g(x, y)$ is the matrix obtained when we completely homogenise $f(x, 1)$ by the introduction of the new variable y, the matrix $g(x, y)$ cannot have any irresoluble divisor other than y, and the highest common factors of the simple minor determinants of $f(x, y)$ and $g(x, y)$ can clearly only differ by a factor which is a power of y. Thus A can have no irresoluble divisor other than y; and similarly, by considering the matrix $f(1, y)$, we can show that A can have no irresoluble divisor other than x. Consequently A has no irresoluble divisor, and must be impotent.

From (C) and (C′) it follows that:

If there are only two variables x and y, and if A is an undegenerate matrix whose elements are homogeneous rational integral functions of x and y all having the same degree, then A is primitive when and only when it is impotent. (D)

Excluding the particular cases considered in the theorems (B), (B'), (C), (C') we can show that:

An undegenerate matrix whose elements are rational integral functions of more than one variable can in general be primitive when it is not impotent and impotent when it is not primitive. (E)

An undegenerate matrix which is impotent but not primitive has been given in the proof of the theorem (A). Illustrations of undegenerate matrices which are primitive but not impotent are given in Exs. *i* and *ii* below.

Ex. *i.* *If* $\phi = \begin{bmatrix} x & y & 0 & 0 \\ 0 & z & y & 0 \\ 0 & 0 & y & x \end{bmatrix}, \qquad \psi = \begin{bmatrix} -y^2 \\ xy \\ -xz \\ yz \end{bmatrix}$

where x, y, z are independent variables, then the matrix ψ is primitive but not impotent.

The affected simple minor determinants of ϕ are the elements of ψ multiplied by x. Therefore ϕ is an undegenerate matrix whose simple minor determinants have the common factor x; aud it is not impotent.

If ϕ is not primitive, there must exist a non-zero one-rowed matrix $[a\ b\ c\ d]$ with constant elements which is connected with the horizontal rows of ϕ, and we must have $[a\ b\ c\ d]\ \psi = 0$,

i, e. $$ay^2 - bxy + cxz - dyz = 0 \qquad (4)$$

This however is impossible; for if there exists an identity of the form (4) in which a, b, c, d, are constants, we must have $a=b=c=d=0$. Consequently ϕ is primitive.

Ex. *ii.* *The matrix* $\phi = \begin{bmatrix} x & y & 0 & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & y & x \end{bmatrix}$

is primitive but not impotent

This matrix is derived from the matrix ϕ of Ex. *i* by the substitution $z=1$. Therefore by Art. 12 it is primitive. It is not impotent because the H. C. F. of its simple minor determinants is x,

Ex. iii. *If* $A = [\,a\,]_m^{m+1}$ *is an undegenerate rational integral functional matrix of rank m in which the number of short rows is greater by one than the number of long rows, then :*

(1) *No simple minor determinant of A can have rank less than* $m-1$.

(2) *If any one simple minor determinant of A vanishes identically, A cannot in general be either primitive or impotent.*

If the minor $[\,a\,]_m^m$ has rank less than $m-1$, then all its minor determinants of order $m-1$ vanish identically, and therefore every minor determinant of A of order m (which can be expanded in terms of these) vanishes identically, which is contrary to the supposition that A has rank m. Thus the first result is true.

To prove the second two results it will be sufficient to consider the case in which the leading simple minor determinant $(a)_m^m$ vanishes identically.

If $(a)_m^m=0$, there exist functions $\lambda_1,\ \lambda_2,\ \ldots\ \lambda_m$, not all vanishing identically, such that $[\,\lambda\,]_m\,[\,a\,]_m^m=0$, and therefore

$$[\,\lambda_1\,\lambda_2\ldots\lambda_m\,]\ \ [\,a\,]_m^{m+1}=[\,0\;0\ldots 0\;\mu\,]=\mu\,[0\;0\ldots 0\;1],$$

where μ is a function which does not vanish identically. Thus the non-zero one-rowed matrix $[0\;0\ \ldots\ 0\;1]$ of degree 0 is connected with the horizontal rows of A, and 0 is one of the horizontal primitive degrees of A. It follows in general that A is not primitive, this conclusion failing only when A has horizontal rows of zero degree with which the matrix $[\,0\;0\ldots 0\;\;1]$ is connected.

Again if the minor determinant $(a)_m^m$ vanishes identically, the minor $[a]_m^m$ of A has rank $m-1$; therefore its reciprocal $[A]_m^m$ can be expressed in the form

$$[\,A\,]_m^m=\begin{bmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_m\end{bmatrix}[\psi_1\;\psi_2\ldots\psi_m].$$

The remaining simple minor determinants of A are then $\Delta_1, \Delta_2, \dots \Delta_m$ given by

$$\Delta_i = a_{1,m+1} A_{1i} + a_{2,m+1} A_{2i} + \dots + a_{m,m+1} A_{mi} = D\,\psi_i,$$

where $$D = a_{1,m+1}\,\phi_1 + a_{2,m+1}\,\phi_2 + \dots + a_{m,m+1}\,\phi_m.$$

Accordingly all the simple minor determinants of A have the common factor D (which does not vanish identically), and we can conclude in general that A is not impotent. This conclusion fails only when D is a constant. It is certainly valid when none of the elements of A contain constant terms, as when A is homogeneous of degree not less than 1.

Ex. *iv. If the matrix $A = [\,a\,]_m^{m+1}$ has rank m, and if all its elements are homogeneous linear functions of the two variables x and y, then it is primitive when and only when it is impotent.*

This theorem is a particular case of (C) and of (D), but we give here an independent proof. If A is impotent, then by (B) it is primitive. It remains to show that if it is not impotent, then it is not primitive.

Let $b_1, b_2, \dots b_{m+1}$, be the affected simple minor determinants of A formed by striking out its 1st, 2nd, .., $(m+1)$th vertical rows, so that

$$[\,a\,]_m \; \overline{\underline{b}}_{m+1} = 0.$$

Then $[\,b\,]_{m+1}$ is an undegenerate matrix normal to A, and $[k_1\, k_2\, \dots k_{m+1}]$ is a one-rowed matrix connected with the horizontal rows of A when and only when

$$[\,k\,]_{m+1} \; \overline{\underline{b}}_{m+1} = 0. \tag{5}$$

If we can determine *constants* $k_1, k_2, \dots k_{m+1}$ such that the equation (5) is satisfied identically, then A is not primitive. Now when A is not impotent, the b's have a common factor ω of degree r, where $r \nless 1$, and we can write $[\,b\,]_{m+1} = \omega\,[\beta]_{m+1}$, where $[\beta]_{m+1}$ is homogeneous of degree $m-r$ in x and y, and replace the equation (5) by

$$k_1\,\beta_1 + k_2\,\beta_2 + \dots + k_{m+1}\,\beta_{m+1} = 0.$$

The condition that this equation shall be an identity when the k's are constants furnishes at most $m-r+1$ homogeneous linear equations to be satisfied by the $m-1$ constants $k_1, k_2, \dots k_{m+1}$, and these always

admit of non-zero solutions. Thus when A is not impotent, we can always determine a non-zero matrix $[k]_{m+1}$ with constant elements which is connected with the horizontal rows of A, i. e., A is not primitive.

In the particular case when one of the simple minor determinants of A vanishes, it follows from Ex. iii. that A is not primitive. This particular case is included in the above proof, for by Ex. iii. it is a case in which A is not impotent.

15. Relations between primitive degrees and minimum degrees of connection.

If $A=[a]_r^n$ and $B=[b]_s^n$ are two rational integral functional matrices of ranks ρ and σ containing the same number of vertical rows, each of them will be said to be *horizontally normal* to the other when

$$[a]_r^n \lceil b \rfloor_n^s = [b]_s^n \lceil a \rfloor_n^r = 0, \quad \text{and} \quad \rho+\sigma=n. \tag{1}$$

In this case every horizontal row of each matrix is orthogonal with every horizontal row of the other matrix, and any one-rowed matrix $[x]_n$ is connected with the horizontal rows of A (or B) when and only when it is orthogonal with every horizontal row of B (or A).

Again if $A=[a]_m^r$ and $B=[b]_m^s$ are two rational integral functional matrices of ranks ρ and σ containing the same number of horizontal rows, each of them will be said to be *vertically normal* to the other when

$$\lceil a \rfloor_r^m [b]_m^s = \lceil b \rfloor_s^m [a]_m^r = 0, \quad \text{and} \quad \rho+\sigma=m. \tag{2}$$

In this case every vertical row of each matrix is orthogonal with every vertical row of the other matrix, and any one-rowed matrix $\lceil x \rfloor_m$ is connected with the vertical rows of A (or B) when and only when it is orthogonal with every vertical row of B (or A).

When any matrix is given, we can always determine a matrix (and if we please an undegenerate matrix) horizontally normal to it, and we can always determine a matrix (and if we please an undegenerate matrix) vertically normal to it. All matrices horizontally normal to a given matrix are horizontally equivalent to one another, and all matrices vertically normal to a given matrix are vertically equivalent to one another.

From the usual definitions of minimum degrees of connection it follows that in (2) the minimum degrees of horizontal connection of A are the horizontal primitive degrees of $\overline{\underline{b}}{}^{m}_{s}$, which are the vertical primitive degrees of B, and that in (1) the minimum degrees of vertical connection of A are the vertical primitive degrees of $\overline{\underline{b}}{}^{s}_{n}$, which are the horizontal primitive degrees of B. Accordingly we have the following two theorems :

The minimum degrees of horizontal connection of any matrix A are the vertical primitive degrees of any matrix B which is vertically normal to A. (A)

The minimum degrees of vertical connection of any matrix A are the horizontal primitive degrees of any matrix B which is horizontally normal to A. (A′)

The same theorems expressed in another form are:

If two matrices are vertically normal to one another, the vertical primitive degrees of either one are the minimum degrees of horizontal connection of the other.

If two matrices are horizontally normal to one another, the horizontal primitive degrees of either one are the minimum degrees of vertical connection of the other.

Ex. *i.* If $A = [\,a\,]_m^n$ is a rational integral functional matrix of rank r, then A has

r horizontal primitive degrees, $m-r$ minimum degrees of horizontal connnection,

r vertical primitive degrees, $n-r$ minimum degrees of vertical connection.

These $m+n$ integers are invariant in all equigradent transformations of A, and in all ordinary linear transformations of the variables.

Ex. *ii.* Since $[1]_m^m\,[0]_m^n = 0$, $[0]_m^n\,[1]_n^n = 0$, the zero matrix $[0]_m^n$ has

no horizontal or vertical primitive degrees,

m minimum degrees of horizontal connection each equal to 0,

n minimum degrees of vertical connection each equal to 0.

Ex. *iii*. If $A = [a]_m^m$ is an undegenerate square matrix, then A has

m horizontal primitive degrees each equal to 0,
m vertical primitive degrees each equal to 0,
no minimnm degrees of horizontal or vertical connection.

The matrix $[0]_m^m$ which is horizontally and vertically normal to A has

m minimum degrees of vertical connection each equal to 0,
m minimum degrees of horizontal connection each equal to 0,
no horizontal or vertical primitive degrees.

Ex. *iv*. If $A=[a]_r^n$ is an undegenerate matrix of rank r, then A has

r vertical primitive degrees each equal to 0,
no minimum degrees of horizontal connection.

The matrix $[0]_r^r$ vertically normal to A has

r minimum degrees of horizontal connection each equal to 0,
no vertical primitive degrees

Ex. *v*. The matrices $\phi=\begin{bmatrix} x & y & 0 & 0 \\ 0 & z & y & 0 \\ 0 & 0 & y & x \end{bmatrix}$, $\psi=[-y^2, \ xy, \ -xz, \ yz]$

are horizontally normal.

The matrix ψ has no vertical connection of degree 0, and the matrix ϕ has no horizontal primitive degree equal to 0.

Ex. *vi*. *If two rational integral functional matrices $\phi = [\phi]_r^n$ and $\psi=[\psi]_s^n$ are horizontally normal to one another, and if the horizontal primitive degrees of one are all 0's, then the horizontal primitive degrees of the other are all 0's.*

Let the ranks of ϕ and ψ be ρ and σ, so that $\rho+\sigma=n$, and let the horizontal primitive degrees of ϕ be all 0's; also let $A=[a]_\rho^n$ be a

horizontally primitive matrix horizontally equivalent to ϕ, all its elements being necessarily constants. Then we can determine an undegenerate matrix $B=[b]_{\sigma}^{n}$ of rank σ with constant elements such that the vertical rows of $\overline{\underline{b}}_{n}^{\sigma}$ are a complete set of σ unconnected solutions of the equation.

$$[a]_{\rho}^{n}\ \overline{\underline{u}}_{n}=0\,;$$

and since B is horizontally normal to A, it is horizontally normal to ϕ, and therefore horizontally equivalent to ψ. Thus there is an undegenerate matrix B with constant elements horizontally equivalent to ψ, i. e., the horizontal primitive degrees of ψ are all 0's.

Ex. *vii. The horizontal primitive degrees of a matrix are all 0's when and only when its minimum degrees of vertical connection are all 0's.*

This follows immediately from Ex. vi.

On the failure of Poisson's equation and of Petrini's generalization.

BY GANESH PRASAD.

The object of this paper is (1) to point out some typical volume distributions for which Poisson's equation is invalid and (2) to prove that Professor Petrini's generalization* of Poisson's equation does not hold for *every* one of these distributions. It is believed that the limited scope of the validity of Professor Petrini's generalization has not been pointed out by any previous writer. I conclude the paper by pointing out the source of error in Professor Petrini's investigation.

1. Let ρ_0 denote the density of the solid at any point P (x, y, z) inside it. Then it should be noted that Poisson's equation fails at P when ∇^2 V is either meaningless or has a value different from $-4\pi\rho_0$, V being the potential due to a small sphere of radius a and centre P.

Some typical cases in which Poisson's equation fails.

2. *Case I.*§ Let the density of the sphere at any point Q (ξ, η, ζ) inside it be

$$\rho = \frac{\cos^2\theta}{\log\frac{1}{r}}$$

where θ is the angle made by P Q with the axis of x and r denotes the distance between P and Q. Then, remembering that the external and internal potentials due to a spherical shell of radius t and surface density P_n $(\cos\theta)$ are

$$\frac{4\pi t P_n}{2n+1} \cdot \left(\frac{t}{r}\right)^{n+1} \text{ and } \frac{4\pi t P_n}{2n+1}\left(\frac{r}{t}\right)^n$$

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* H. Petrini, "Les dérivées premières et secondes du potentiel," *Acta Mathematica*, t. 31, p. 182, 1908.

§ This case has been studied by Prof. Petrini (*loc. cit.*, p. 186), but the treatment given by me is different from his.

respectively, we have the potential at Q given by

$$V=\int_0^r \left\{ \frac{4\pi t}{3} \cdot \frac{t}{r} + \frac{8\pi t}{15} P_2 \cdot \left(\frac{t}{r} \right)^3 \right\} \frac{dt}{\log \frac{1}{t}}$$

$$+\int_r^a \left\{ \frac{4\pi t}{3} + \frac{8\pi t}{15} P_2 \cdot \left(\frac{r}{t} \right)^2 \right\} \frac{dt}{\log \frac{1}{t}} .$$

Hence

$$\frac{\delta V(\xi, \eta, \zeta)}{\delta r} = -\int_0^r \left\{ \frac{4\pi t^2}{3r^2} + \frac{8\pi t^4}{5r^4} P_2 \right\} \frac{dt}{\log \frac{1}{t}}$$

$$+ \frac{16\pi r P_2}{15} \int_r^a \frac{dt}{t \log \frac{1}{t}} = \frac{16\pi r}{15} P_2 \log \left\{ \frac{\log \frac{1}{r}}{\log \frac{1}{a}} \right\}$$

$$-\frac{4\pi}{3r^2} \left\{ k_1 \cdot \frac{r^3}{\log \frac{1}{r}} \right\} - \frac{8\pi}{5r^4} P_2 \left\{ k_2 \cdot \frac{r^5}{\log \frac{1}{r}} \right\}$$

where k_1 and k_2 are proper fractions dependent on r.

Therefore

$$\frac{1}{r} \frac{\delta V(\xi, \eta, \zeta)}{\delta r} = \frac{16\pi}{15} P_2 \cdot \log \frac{\log \frac{1}{r}}{\log \frac{1}{a}} - \frac{4\pi k_1}{3 \log \frac{1}{r}} - \frac{8\pi k_2}{5 \log \frac{1}{r}} P_2 \quad \text{(A}$$

From the above equation it follows at once that since the first differential coefficients of V are all zero at P,

$$\frac{\delta^2 V}{\delta x^2} = \lim_{h=0} \frac{1}{h} \frac{\delta V(x+h, y, z)}{\delta x} = +\infty,$$

$$\frac{\delta^2 V}{\delta y^2} = \lim_{h=0} \frac{1}{h} \frac{\delta V(x, y+h, z)}{\delta y} = -\infty,$$

$$\frac{\delta^2 V}{\delta z^2} = \lim_{h=0} \frac{1}{h} \frac{\delta V(x, y, z+h)}{\delta z} = -\infty.$$

Thus $\nabla^2 V$ has no meaning and, consequently, Poisson's equation fails at P.

3. *Case II.** Let $\rho = \cos\left(\log \frac{1}{r}\right)$. Then by Newton's theorem relating to the attraction of a spherical shell, at P

$$\frac{\delta V}{\delta r} = 0 \text{ and at Q}$$

$$\frac{\delta V}{\delta r} = -\frac{4\pi \int_0^r t^2 \cos\left(\log \frac{1}{t}\right) dt}{r^2} \qquad (1)$$

Now, putting $t = e^{-v}$, we have

$$\int_0^r t^2 \cos\left(\log \frac{1}{t}\right) dt = \int_{\log \frac{1}{r}}^{\infty} e^{-3v} \cos v \, dv$$

$$= \frac{r^3}{\sqrt{10}} \left\{ \cos\left(\log \frac{1}{r} + \tan^{-1} \tfrac{1}{3}\right) \right\}.$$

Therefore, from (1),

$$\frac{1}{r} \frac{\delta V(\xi, \eta, \zeta)}{\delta r} = -\frac{4\pi}{\sqrt{10}} \left\{ \cos\left(\log \frac{1}{r} + \tan^{-1} \tfrac{1}{3}\right) \right\}. \qquad (B)$$

Hence it follows at once that $\frac{\delta^2 V}{\delta x^2}$, $\frac{\delta^2 V}{\delta y^2}$, $\frac{\delta^2 V}{\delta z^2}$, are all non-existent. Thus $\nabla^2 V$ has no meaning and, consequently, Poisson's equation fails at P.

4. *Case III.* Let $\rho = \cos \frac{1}{r}$. Then, proceeding as in Case II, we find that

$$\frac{1}{r} \frac{\delta V(\xi, \eta, \zeta)}{\delta r} = \frac{-4\pi \int_0^r t^2 \cos \frac{1}{t} \, dt}{r^3} \qquad (2)$$

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* For Cases II and III see my paper. "On the second derivates of the Newtonian potential due to a volume distribution having a discontinuity of the second kind" (*Bulletin of the Calcutta Mathematical Society*, Vol. 6, 1916,) in which are studied, for the first time, volume distributions for which $\lim_{r=0} \rho$ is non-existent.

But, putting $\frac{1}{t} = v$; we have

$$\int_{o}^{r} t^{2} \cos\frac{1}{t}\, dt = \int_{\frac{1}{r}}^{\infty} \frac{\cos v}{v^{4}}\, dv$$

which is numerically less than $2r^{4}$, since $\frac{1}{v^{4}}$ is always positive and constantly diminishes as v increases.

Therefore, from (2),

$$\left|\frac{1}{r} \quad \frac{\delta V}{\delta r}\right| < 8r\pi..$$

Hence it follows at once that $\frac{\delta^2 V}{\delta x^2}, \frac{\delta^2 V}{\delta y^2}, \frac{\delta^2 V}{\delta z^2}$, are all zero. Thus $\nabla^2 V$ is zero and, consequently, Poisson's equation fails at P unless we assign the value 0 to ρ_0. It should be noted that, in this case, ρ_0 may be assigned *any* value without affecting the valve of $\nabla^2 V$.

Petrini's generalization of Poisson's equation.

5. Professor Petrini has formulated the following generalization of Poisson's equation: "La fonction ΔV existe toujours, même si les dérivées $\frac{\delta^2 V}{\delta x^2}$, $\frac{\delta^2 V}{\delta y^2}$ et $\frac{\delta^2 V}{\delta z^2}$ n' existent pas séparément, si on définit

$$\Delta V = \lim_{\substack{h_1 = o,\\ h_2 = o,\\ h_3 = o}} \sum_{x,y,z} \frac{1}{h_1}\left[\frac{\delta V\,(x+h_1, y, z)}{\delta x} - \frac{\delta V\,(x, y, z)}{\delta x}\right], \text{ où } \lim \frac{h_\lambda}{h_\mu} \neq o \text{ et déterminée."}$$

It is easily seen from (A) of Art. 2 that the generalizatian holds for the Case I. For, let

$$\frac{h_1}{a} = \frac{h_2}{\beta} = h_3,$$

where α, β are always different from zero as well as from infinity. Then

$$\Delta V = \lim_{h_3=0} \left\{ + \frac{16\pi}{15} \log \frac{\log \frac{1}{h_3 \alpha}}{\log \frac{1}{a}} - \frac{8\pi}{15} \log \frac{\log \frac{1}{h_3 \beta}}{\log \frac{1}{a}} - \frac{8\pi}{15} \log \frac{\log \frac{1}{h_3}}{\log \frac{1}{a}} \right\}.$$

Now the expression within the crooked brackets is equal to

$$\frac{8\pi}{15} \log \frac{\left\{ \log \left(\frac{1}{\alpha h_3} \right) \right\}^2}{\left\{ \log \frac{1}{h_3} \right\} \left\{ \log \frac{1}{h_3} + \log \frac{1}{\beta} \right\}} = \frac{8\pi}{15} \log 1$$

in the limit.

Thus $\Delta V = 0 = -4\pi \rho_0$.

Failure of Petrini's generalization.

6. It is easily seen from (B) that Professor Petrini's generalization fails for the Case II. For,

$$\Delta V = \lim_{h_3=0} \left[\frac{-4\pi}{\sqrt{10}} \cdot \left\{ \cos \left(\log \frac{1}{h_3} + \log \frac{1}{\alpha} + \tan^{-1} \frac{1}{3} \right) + \cos \left(\log \frac{1}{h_3} + \log \frac{1}{\beta} + \tan^{-1} \frac{1}{3} \right) + \cos \left(\log \frac{1}{h_3} + \tan^{-1} \frac{1}{3} \right) \right\} \right]$$

which exists only for *special* values of α, β. In fact the necessary conditions for the existence of ΔV are the following:—

$$1 + \cos \left(\log \frac{1}{\alpha} \right) + \cos \left(\log \frac{1}{\beta} \right) = 0 \text{ in the limit;}$$

$$\sin \left(\log \frac{1}{\alpha} \right) + \sin \left(\log \frac{1}{\beta} \right) = 0 \text{ in the limit.}$$

When these conditions are satisfied $\Delta V = 0$. Thus ΔV exists (and is zero) *only* when, in the limit, $\frac{h_3}{h_1}$ and $\frac{h_3}{h_2}$ are of the forms

$$e^{\left(2m\pi \pm \frac{2\pi}{3}\right)} \quad \text{and} \quad e^{\left(2n\pi \mp \frac{2\pi}{3}\right)}$$

respectively, m, and n being any integers.

Source of error in Petrini's investigation.

7. It is clear from the preceding articles that Professor Petrini's results are not *necessarily* true when ρ has a discontinuity of the *second* kind. But this fact seems to have been overlooked both by Professor Petrini, who claims* his investigation to be the most comprehensive possible, and by Professor Wangerin who accepts this claim without any criticism§. I proceed to point out the chief source of error in Professor Petrini's memoir.

8. Let $\xi=x+r\cos\theta,\ \eta=y+r\sin\theta\cos\psi, \zeta=z+r\sin\theta\sin\psi$; also let

$$\cos\theta=u,\ F(\xi,\eta,\zeta)=\frac{\delta}{\delta\xi}V(\xi,\eta,\zeta).$$

Then

$$F(x,y,z)=\int_0^{2\pi}d\psi\int_{-1}^{1}u\,du\int_0^a\rho\,dr,$$

$$F(x+h,y,z)=\int_0^{2\pi}d\psi\int_{-1}^{1}du\int_0^a\rho\,\frac{ru-h}{R^3}r^2dr,$$

where $R=\sqrt{r^2-2rhu+h^2}$.

Putting $r=hp$, $q=\sqrt{p^2-2pu+1}$, we have

$$\frac{1}{h}\Big\{F(x+h,y,z)-F(x,y,z)\Big\}$$

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* "Dans le présent mémoire j'ai essayé de donner les conditions nécessaires et suffisantes pour l'existence des dérivées premieres et secondes du potentiel en ne faisant en général sur la densité d'autre hypothése dispensable que celle qu'elle est finie" (*loc. cit*, p. 129).

§ "In der sehr umfangreichen Abhandlung, von der einige Resultate bereits früher veroffentlicht sind, stellt sich der Verf. die Aufgabe, die notwendigen und hinreichenden Bedingungen für die Existenz der zweiten Ableitungen des Körperpotentials, der ersten und zweiten Ableitungen des Flächenpotentials sowie der ersten Ableitungen des Potentials einer Doppelschicht zu ermitteln, ohne über die Dichtigkeit andere Voraussetzungen zu machen, als dass sie endlich ist" (See Prof. Wangerin's review of Prof. Petrini's memoir in the *Fortschritte der Mathematik, Bd. 39*, pp. 819—823.)

$$=\int_0^{2\pi} d\psi \int_{-1}^{1} du \int_0^{1} \rho \left(\frac{pu-1}{q^3}\, p^2 - u\right) dp$$

$$+\int_0^{2\pi} d\psi \int_{-1}^{1} du \int_1^{\frac{a}{h}} \rho \left(\frac{pu-1}{q^3}\, p^2 - u\right) dp.$$

Now, denoting the first triple integral by I_h, Prof. Petrini correctly says that

$$I_h = \rho' \int_0^{2\pi} d\psi \int_0^{1} p^2 dp \int_{-1}^{1} \frac{\delta}{\delta u}\left(\frac{p-u}{q}\right) du - \int_0^{2\pi} d\psi \int_0^{1} u\, du \int_0^{1} \rho\, dp$$

where ρ' is a mean value of ρ.

But he makes a mistake in concluding that

"$\underset{h=0}{\text{Lim}}\ I_h$ est finie",

for the limit may not at all exist when ρ does not tend to a limit as (ξ, η, ζ) tends to (x, y, z).

9. The above-mentioned mistake of Prof. Petrini renders defective and misleading not only his generalization of Poisson's equation and his condition for the existence of $\frac{\delta^2 V}{\delta x^2}$ but also a number of minor results* scattered throughout the first chapter of the memoir.

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* *e.g.* "Si ρ est fonction de r seulement, la dérivée $\frac{\delta^2 V}{\delta x^2}$ existe dans tout l'espace; elle est aussi continue dans tous les points où la fonction ρ est continue" (Art 4, p. 139). This statement of Prof. Petrini is incorrect for the case $\rho = \cos \log \frac{1}{r}$.

On an interpretation of Fermat's Law.

By C. V. Raman.

The *Philosophical Magazine* for July 1913 contains a mathematical paper by Dr. D. N. Mallik on "Fermat's Law" which appears to call for comment. The part of this paper which purports to be an original contribution to optical theory is very brief and may be reproduced here, the rest of the paper being a superstructure built up of various well-known results obtained by Larmor, Poincaré, J. J. Thomson and others. Dr. Mallik writes Fermat's Law of earliest arrival in the form

$$\delta \int dt = o \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

where δ is the operator of the Calculus of Variations, and proposes to interpret it in the following manner by utilizing Hamilton's well-known dynamical principle. Says Dr. Mallik, "The configuration of equilibrium and motion of a dynamical system is defined by

$$\delta \int (\mathrm{T}-\mathrm{V})\, dt = o \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2)$$

where T=kinetic energy and V=Potential energy. If this is to be consistent with Fermat's Law we must have, for light propagation,"

$$\mathrm{T}-\mathrm{V}=\mathrm{C} \text{ (Constant)} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3)$$

"Again, from the principle of energy

$$\mathrm{T}+\mathrm{V}=\mathrm{C}' \text{ (Constant)} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4)$$

From (3) and (4) we get"

$$\left.\begin{array}{l} 2\ \mathrm{T}=\mathrm{C}'+\mathrm{C} \\ 2\ \mathrm{V}=\mathrm{C}'-\mathrm{C} \end{array}\right\} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (5)$$

Dr. Mallik recognizes at this stage that there is a certain absurdity in the result given in (5) at which he has arrived, for he remarks "But this is meaningless since the *mean* potential energy and the mean kinetic energy are alone constant, as these quantities are understood to mean in the above equations. Accordingly, the only conclusion that seems to be consistent with all the equations is that *the optical energy is entirely kinetic.*"

Dr. Mallik's remark quoted above is evidently wide of the mark, in as much as T and V used in equations (2) and (4) certainly do *not* refer to the *mean* kinetic and potential energies

respectively. The real absurdity is to be looked for in the remarkable result T—V=Constant, contained in (3) above. which is obtained by Dr. Mallik and is stated by him to be a necessary consequence of Fermat's law, (1) and Hamilton's Principle (3) taken together. I fear that this result, considered as a proposition in analysis will not stand the slightest scrutiny. In the first place, t in Hamilton's, Principle is considered as an independent variable, whereas in Fermat's Law as stated by Dr. Mallik, it is evidently a *dependent* variable determined by the path over which the integration $\int \mu ds$ is effected. Even if we ignore this point, it does not at all follow that the various values of F (x) which satisfy the relation

$$\delta \int F(x, y, \dot{y})\, dx = 0 \qquad \ldots \qquad \ldots \qquad \ldots \quad (6)$$

are necessarily proportional to one another. This is what Dr. Mallik tacitly assumes to be the case, and it may readily be shown by a few actual cases that such an assertion would be quite baseless. For example, take

$$F(x, y, \dot{y}) = x^2 + y^2 + 2\, a\, y\, \dot{y} \qquad \ldots \qquad \ldots \qquad \ldots \quad (7)$$

To satisfy (6), we must, by a well-known theorem in the Calculus of Variations, find that

$$\frac{dF}{dy} = \frac{d}{dx} \frac{dF}{d\dot{y}} \qquad \ldots \qquad \ldots \qquad \ldots \quad (8)$$

From (7) and (8), we have

$$2y = \frac{d}{dx}(2ay)$$

$$\therefore \quad a \frac{dy}{dx} = y,$$

$$\text{and } a \log \frac{y}{a} = x$$

$$\text{and } y = a\, e^{\frac{x}{a}}, \qquad a \text{ being a constant.}$$

$$\therefore \quad F(x) = x^2 + 3\, a^2\, e^{\frac{2x}{a}} \qquad \ldots \qquad \ldots \quad (9)$$

It is obvious that if we substitute various values a_1, a_2, etc. for a in (9), the different values of F (x) thus obtained would not be proportional to one another.

Dr. Mallik's proposed interpretation of Fermat's Law is thus without a proper foundation in analysis. As I shall presently show, it also appears to involve a certain confusion in fundamental physical ideas. For, if we commence with a consideration of some consistent system to which the ordinary equations of dynamics may be applied, and postulate the usual stress-strain relations, e.g., those of an elastic solid or those of a compressible fluid, the ordinary equations of wave-propagation are obtained, and taking the wave-normals to be the rays in accordance with Huyghens' Principle, Fermat's Law follows at once from elementary geometry. Hamilton's Principle is a statement in special form of the ordinary equations of dynamics, and since Fermat's Law is also to be arrived at from these laws without any assumptions regarding the *intrinsic* nature of the stress or strain of the medium, it is obviously impossible by combining Fermat's Law and Hamilton's Principle in the manner proposed by Dr. Mallik to get behind the fundamental stress-strain relations postulated as a basis for forming the equations of wave-propagation. Such attempts would be fruitful of errors in analysis.

Perhaps the most remarkable part of Dr. Mallik's paper is that in which he remarks, "But, if the interpretation of Fermat's Law sketched out above is admissible, we are led to a further generalization—* * * * ." The attitude of conscious doubt regarding the correctness of the interpretation indicated in this passage, seems difficult to reconcile with the language of rigour used by Dr. Mallik in sketching out the main steps of his analysis as reproduced above. Perhaps Dr. Mallik was himself less convinced of the soundness of his own work than might well be expected of a mathematician publishing a paper in the *Philosophical Magazine.*

II

Dr. Mallik has written a note that appears later in this issue and that I have been enabled to see in proof by courtesy of the Honorary Secretary. In it, he does not appear to make any attempt to find a mathematical or physical justification for the propositions put forward by him and criticized by me. Perhaps it would have been more graceful if Dr. Mallik had frankly acknowledged the correctness of my criticisms.

Dr. Mallik's paper rests entirely upon his statement "*we must have* for light propogation, T—V = constant," in order that Hamilton's Principle and Fermats' Law might be consistent with each other.

That this statement of Dr. Mallik is most seriously in error may be shown in a very simple and general manner. In the equation

$$\delta \int (T-V)\, dt = 0,$$

the operator δ signifies the effect on the value of the integral of a variation in the *functional* relationship between t, and the variables involved in $(T-V)$. If, therefore, we add to $(T-V)$, any *known* function of the time, say a sin nt, we should still find

$$\delta \int \left[(T-V) + a \textit{ Sin } nt \right] dt = 0,$$

the term added within the sign of integration contributing nothing to the variation. If Dr. Mallik is entitled to argue that $(T-V)$ is a constant, it could equally well be argued that $(T-V) + a$ *Sin* nt is constant. This *reductio ad absurdum* demonstrates the fallacy of Dr. Mallik's contention.

As Dr. Mallik's analysis is demonstrably incorrect, his physics which depends on the analysis for its foundation naturally ceases to have any value or significance. The inconsistencies and inaccuracies contained in his note well illustrate the confusion of ideas on which I have already commented. For instance, in the penultimate para of his note (as seen by me in proof), he refers to his trying to analyse the intimate nature of a medium like the ether, while in the para just preceding he denies making any such attempt, and even denies any relation between Huyghens' Principle and the equations of wave-motion. I would suggest that Dr. Mallik should read Huyghens' own statement of the case in his original treatise on Light. Dr. Mallik might possibly also obtain some clearer ideas on the relation between Hamilton's Principle and Optical theory by reading Wangerin's exposition of the work of Voigt on the subject. [Encyclopædie Der Math. Wiss., Band V, Art. 39.] From the authoritative reference quoted above, it will be seen that Dr. Mallik is in error when he assumes (without any analytical justification) that $T - V =$ constant for an optical medium. Such an assumption is wholly unnecessary and leads to results which are quite meaningless.

On the vibrations of a membrane bounded by two non-concentric circles.

By Sudhansukumar Banerji.

In this paper I propose to give the complete solution of the problem of the vibrations of a membrane bounded by two circles which are not concentric. This problem remained unsolved up to this time although the corresponding problem for the case in which the circles are concentric is well-known to be easy of solution.

The method adopted is one of continued approximations and is analogous to the method used in two previous papers published by me in this Bulletin.*

I should like to express my indebtedness to Dr. Ganesh Prasad under whom I carried on the investigation.

§ I.

Suppose that a point P has the coordinates (r, θ) and (r', θ') referred to two points A and B, θ and θ' being measured in opposite senses from the line AB, and let AB$=\delta$ so that

$$r^2 = r'^2 + \delta^2 - 2r'\delta \cos \theta'.$$

Then the following theorems† hold true :—

$$\text{(I)} \quad (-1)^n J_n(r) \cos n\theta = \sum_{p=-\infty}^{p=\infty} (-1)^p J_p(\delta)\, J_{n-p}(r') \cos(n-p)\theta'.$$

$$\text{(II)} \quad (-1)^n D_n(r) \cos n\theta = \sum_{p=-\infty}^{p=\infty} (-1)^p J_p(\delta)\, D_{n-p}(r') \cos(n-p)\theta' \qquad (\text{if } r' > \delta),$$

$$\text{or} \quad \sum_{p=-\infty}^{p=\infty} (-1)^p D_p(\delta)\, J_{n-p}(r') \cos(n-p)\theta' \qquad (\text{if } r' < \delta),$$

where $D_n(r) = r^n \left(-\dfrac{d}{rdr} \right)^n D_0(r)$, $\quad D_0(r) = \dfrac{2}{\pi} \displaystyle\int_0^\infty e^{-ir\cosh u}\, du$.

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* See Vols. 4 and 5.

† See Graf and Gubler · *Einleitung in die Theorie der Bessel' schen Funktionen*, Ch. X, Art 5, pp. 81-84, also Prof. Graf's paper in *Math. Ann.*, Vol. 43.

Similar theorems will also hold true when the cosine is replaced by sine.

§ 2.

Let the membrane be bounded by two non-concentric circles of radii a and b.

Let, as before, (r, θ) be the polar coordinates of a point P inside the two circular boundaries referred to the centre A of the outer circle. Let (r', θ') be the polar coordinates of the same point referred to B, the centre of the inner circle, θ and θ' being measured in opposite senses from the line AB.

If w denote the normal displacement of a point P at any time t, it must satisfy the following conditions:—

(1)

$$\frac{d^2 w}{dt^2} = c^2 \left(\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} \right);$$

(2) it must vanish both on the inner and the outer boundaries;

(3) its first and second differential coefficients must be finite and continuous everywhere inside the two circular boundaries.

Let us first assume for w the following expression:—

$$w = w_0 = J_n (kr) \cos n\theta \, e^{ikct}$$

This satisfies the differential equation (1), and will also satisfy the condition on the outer boundary, provided k is a root of the equation

$$J_n (ka) = 0.$$

But this will not satisfy the condition on the inner boundary.

The value of this function on the inner boundary is given by

$$w_0 = (-1)^n \sum_{p=-\infty}^{p=\infty} (-1)^p J_p (k\delta) \, J_{n-p} (kr') \cos (n-p)\theta' \, e^{ikct},$$

when $r' = b$.

(Theorem I.)

To satisfy the condition on the inner boundary take a second function w_1 given by

$$w_1 = \sum_{p=-\infty}^{p=\infty} \overset{(1)}{A}_{n-p} \, D_{n-p} (kr') \cos (n-p)\theta' . \, e^{ikct},$$

This will make $w_0 + w_1$ vanish on the inner boundary if we take $\overset{(1)}{A}_{n-p}$ to be given by the expression

$$(-1)^{n+p} J_p\,(k\delta)\, J_{n-p}\,(kb) = -\overset{(1)}{A}_{n-p}\, D_{n-p}\,(kb)$$

i.e.,
$$\overset{(1)}{A}_{n-p} = (-)^{n+p+1} J_p\,(k\delta)\,\frac{J_{n-p}(kb)}{D_{n-p}(kb)}.$$

But this will upset the condition on the outer boundary. To satisfy the condition on the outer boundary, introduce a third function w_2 given by

$$w_2 = \sum_{s=-\infty}^{s=\infty} \overset{(2)}{A}_s\, J_s\,(kr) \cos s\theta.\, e^{ikct}.$$

Since near the outer boundary $r > \delta$, we have near the outer boundary

$$w_1 = \sum_{n=-\infty}^{n=\infty} \overset{(1)}{A}_n\, D_n\,(kr') \cos n\theta'.\, e^{ikct}$$

$$= \sum_{n=-\infty}^{n=\infty} \overset{(1)}{A}_n (-1)^n \sum_{p=-\infty}^{p=\infty} (-1)^p J_p\,(k\delta)\, D_{n-p}\,(kr)$$

$$\cos\,(n-p)\theta.\, e^{ikct}.$$

(Theorem II.)

So that if we write $n-p=s$, we see that $w_0 + w_1 + w_2$ will vanish on the outer boundary if $\overset{(2)}{A}_s$ is given by

$$\overset{(2)}{A}_s\, J_s\,(ka) = -\sum_{n=-\infty}^{n=\infty} \overset{(1)}{A}_n (-1)^s J_{n-s}\,(k\delta)\, D_s\,(ka)$$

that is,
$$\overset{(2)}{A}_s = -\frac{D_s(ka)}{J_s\,(ka)}\,(-1)^s \sum_{n=-\infty}^{n=\infty} \overset{(1)}{A}_n\, J_{n-s}(k\delta).$$

This will again upset the condition on the inner boundary. To satisfy this condition we introduce a fourth function w_3 in a similar manner and so on.

The solution of the problem is then given by

$$w = w_0 + w_1 + w_2 + w_3 + \;\ldots\;\;\ldots$$

Reply to Mr. Raman's criticism of Dr. Mallik's paper on Fermat's Law.

By

D. N. Mallik.

In the very first remark which Mr. Raman* makes, he seems to suggest that I have taken T, V in the usual Hamiltonian equation $\delta \int (T-V)\, dt = o$, as *mean kinetic* and *potential* energies. No doubt, it would have been surprising, if I had done so. As a matter of fact, not only have I not done so, but have really based my argument on the fact that they are *not mean* kinetic and potential energies. For I state: These equations $2T = C + C'$, $2V = C' - C'$ are meaningless "since the mean potential energy and mean kinetic energy are alone constant." Surely if these had been taken by me (erroneously) to be mean kinetic and potential energies, the equations would *not* have been meaningless.

Then he declares "the real absurdity is to be looked for in the result $T-V$ is constant" and then proceeds to establish this thesis.

"In the first place, t in Hamilton's principle is considered as an independent variable and in Fermat's Law, it is *evidently* a *dependent* variable." As he does not explain why it is so, and as he after all ignores this point, I shall not follow him further in this regard, for the present.

He then attempts to prove that $F(x,y,\dot{y})$ where $\delta \int F(x,y,\dot{y})\, dx = o$ has certain properties. The relevancy of this is not apparent. For on my notation, according to him $T-V = F(t, y, \dot{y})$ and as a particular case, even $= t^2 + 2ay\dot{y} + y^2$.

Remembering that $T-V$ refers to a vibrating ethereal medium—vibrating about a state of equilibrium, such assumptions seem to require justification which is not forthcoming in the present instance. For myself, I do not know of any theory of the ether on which such a form of $T-V$ may be constructed; nor do I understand how such a form can be consistent with small motion.† Morever, it is not possible to find $F(x)$ or $F(t)$ from the variational equation, for the simple reason that $T-V$ refers to *any* dynamical system arbitrarily chosen.

* See his paper in this issue of the *Bulletin*.

† For this one need only consider $\dfrac{d}{dt}\dfrac{\delta L}{\delta \dot{y}} + \dfrac{\delta L}{\delta y} = o$ where $L = T - V$.

But, apart from that, Mr. Raman's analysis does not concern me for I have not tacitly assumed that "values of $F(x)$ or $F(t)$ derivable from the above are proportional to one another."

My point is simply this: We have the dynamical equation $\delta \int (T-V)\, dt=o$, which gives a *complete* account of the motion of a dynamical system—in this case, the disturbed optical medium.

We have *also* the equation $\delta \int dt=o$.

Are these independent of each other? If so, the second equation can only be *the equation of constraint.* No such constraint can, in the present state of our knowledge, be well associated with the medium considered.

If no such constraint can be postulated, we must regard the second equation as identical with the first in the particular case of optical disturbance and for this $T-V$=constant seems to be the necessary condition, in the most general case as $T-V=f(t)$ is obviously excluded.

Mr. Raman then proceeds to tell me that my physical ideas are somewhat confused. I should hardly wonder at it. I have accordingly tried to understand and carefully follow his exposition. He explains how the ordinary equations of wave propagation are obtained and states that taking wave normals to be the rays in accordance with Huyghens' principle, Fermat's Law follows by elementary geometry. I do not know what the equations of motion have to do in this case, for, any form of wave surface being postulated, Huyghens' principle (which calls for explanation, as much as any other) would yield Fermat's Law. He goes on: "Hamilton's principle is simply a statement in a special form of the ordinary equations of Dynamics." Hamilton's principle is really more than that; it is the single principle which completely represents the motion of a dynamical system, so that any other equation, unless it is that of a constraint, must be derivable from it. Thus, Fermat's principle ought to be so derivable. But Hamilton's principle gives no information regarding the constitution of the medium, for the latter determines the form of T and V, which must be derived from a subsidiary hypothesis. In the same way, Fermat's Law is not competent to give any information regarding "stress-strain relation," etc. For this a subsidiary hypothesis such as Fresnel's, as I remarked in my paper on Fermat's Law is necessary. Neither Hamilton's principle nor Fermat's Law can, therefore, be used to go behind the constitution of the medium. And I have not so used it, either.

Finally, Mr. Raman comments on the tone of diffidence in which I speak of the interpretation of Fermat's Law, which I have given. He demurs to such a position on the part of a mathematician

writing for the *Philosophical Magazine.* The reason is simple. I have proceeded on the supposition that $\delta \int dt = o$ is not an equation of constraint. When however, we are trying to analyse the intimate nature of a medium like the ether, such a supposition can only be made with with due diffidence, and, in view of the remarkable generalisation to which it leads, such diffidence is not only natural but reasonable.

Further, the Hamiltonian principle postulates that the initial and final configuration of a dynamic system are prescribed and the time of transit of the system from the initial to the final configuration must remain unchanged. The same conditions may be imposed on Fermat's principle, if we take the time of transit to be that from one wave-front to the next. In this case, t in both expressions will have the same meaning, but this *may* deprive my conclusions of a part of their generality, in a manner, it is not possible, to decipher at present.

Sophie Kovalevsky—The Great Lady Mathematician.*

(1850-1891)

By

Haripprasanna Banerjea.

§ 1. *Early life.*

1. Sophie Corvin—Kroukovesky was born at Moskow, on the 15th January, 1850. In September, 1860, her husband, Waldemar Kovalevsky, who became later on a distinguished palaeontologist, went with her to Heidelburg. Here she attended lectures for full three terms and studied Mathematics and Physics under Kirchhoff, Königsberger, Du Bois Reymond and Helmholtz. From the age of fifteen, she had applied herself enthusiastically to the study of Mathematics with the result that in 1868 she was quite familiar with the elements of geometry and of the infinitesimal calculus. In 1870, she went to Berlin, where she remained for three years, studying Mathematics under the special direction of Weierstrass.

2. In the course of the summer of 1874, Sophie Kovalevsky was given the degree of the Doctor of Philosophy, by the University of Göttingen, her thesis being the German memoir "About the theory of partial differential equations", published in *Crelle's Journal*, vol. 30. She had also presented to the University two other papers in German, viz.

(1) "On the reduction, to elliptic integrals, of a definite class of Abelian integrals of the third rank",
(published afterwards in *Acta Mathematica*, vol. 4.),

(2) "Additions to, and remarks on, Laplace's investigation on the form of Saturn's rings."
(published in *Astronomische Nachrichten*, vol. 3.).

3. On becoming a doctor, Sophie Kovalevsky returned to Russia. On 5th October, 1878, she gave birth to a daughter. In March, 1883, Waldemar Kovalevsky died at Moskow in tragic circumstances. Sophie Kovalevsky happened to be in Paris at that time. When she

* For the substance of this sketch of Kovalevsky's life, I am indebted to Professor Mittag-Leffler's "Notice biographique" (See *Acta Mathematica*, vol. 16).

heard of the death of her husband and the tragic circumstances which accompained it, she fell ill and lay between life and death for about a month.

4. Towards the end of June, close upon her recovery, she went to rejoin at Berlin her faithful friend and professor, Weierstrass. She resumed with enthusiasm her mathematical researches, and finished a paper in German, which she published under the title "On the refraction of light in crystalline media" (*Acta Mathematica*, vol. 6).

§ 2. *Her career as a University Professor.*

5. Sometime before the death of her husband, Sophie Kovalevsky was very desirous of dedicating herself to the profession of education by becoming a professor in some University. Professor Mittag-Leffler, who was then Professor of Mathematics in the University of Helsingfors, the capital of Finland, had a very high opinion of her talents; so about the autumn of 1880, he tried to take her as his associate Professor. The project failed. But he succeeded in making her his associate, when he was called to the new University at Stockholm in the spring of 1881.

6. For Sophie Kovalevsky herself, the principal difficulty which opposed the realisation of her desire, vanished with the death of her husband. In a letter dated the 5th of August 1883, Weierstrass informed Professor Mittag-Leffler that she was disposed to give a course of lectures in Mathematics at Stockholm, but at the begining she was determined to give to the lectures no public character. In December 1883, Sophie Kavalevsky reached Stockholm, and about the spring of 1884, before a selected but attentive audience, she explained in the German language, the theory of partial differential equations.

7. Thanks to the success of the lectures and to the impression produced on the educated community of Stockholm by the sympathetic and genial personality of the lecturer, it was possible to procure funds to appoint Madame Kovalevsky, professor of higher analysis at the University of Stockholm for a period of five years. Notwithstanding the shortness of the time during which she was lecturer in Stockholm, she had already become sufficiently conversant with the Swedish language, to lecture in it, from the begining of her career as a professor of the University. Before the expiry of her term of office, Sophie Kovalevsky was awarded by the Institute of France, the Bordin prize for her paper "*Sur la probléme de la rotation d'un corps solide autour d'un point fixe.*" (Acta Mathematica, vol. 12, and Mémoires présentés par divers savants á la Academie des Sciences de l' Institut national de France, tome 31.)

8. This circumstance facilitated the efforts made by her admirers to get the necessary funds for establishing definitely the chair of higher analysis at the University of Stockholm, and in the spring of 1889, it was decided to continue the services of Madame Kovalevsky as a University Professor for life.

9. However, she did not enjoy this honour for a long time. Madame Kovalevsky had passed the winter vacation of 1890-91, at Midi, near the Mediterranean sea-coast of France. During the return voyage she caught a chill, and on the 6th February, 1891, after having delivered in the morning her first lecture of the year, she was obliged to take to bed, no more to get up. She died on the 10th February, in the morning, of violent pleuresy which was probably a form of influenza and which from the begining had defied medical skill.

10. Sophie Kovalevsky is best known because of her epoch-making discovery of a new type of the motion of a rigid body about a point. This type of motion is referred to in text-books as *Kovalevsky's top.** The total number of original papers published by her is only ten.

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* See Whittaker's *Analytical Dynamics*, p. 160.

Review.

The mathematical analysis of electrical and optical wave-motion on the basis of Maxwell's equations. By H. Bateman, M.A., Ph.D. Pp. 159. Cambridge University Press, 1915.

This book is intended as an introduction to some recent developments of Maxwell's electro-magnetic theory which are directly connected with the solution of the partial differential equation of wave-motion. The higher developments of the theory which are based on the dynamical equations of motion are not considered at all. Even with this limitation the subject is a vast one, and to bring the work of perusing the literature within his power, the author has omitted an account of the modern theory of relativity which has been expounded very clearly in several recent publications.

The book is divided into nine chapters. Ch. I deals with the fundamental ideas about wave-motion and Ch. II gives a general survey of the different methods of solving the wave-equation, *viz.*

$$\nabla^2 V + k^2 V = o.$$

Ch. III deals with the solutions in polar co-ordinates. After enumerating all the important solutions, and the relations between various solutions, the author considers the convergency of the series occurring in the solutions. Next the author applies the results to the solution of certain problems, including (1) the scattering and the absorption of waves by a spherical obstacle, and (2) the pressure of radiation on a spherical obstacle. Ch. IV treats of the solutions of the wave-equation in cylindrical co-ordinates and of their applications to several problems, including the propagation of electric waves (1) on a semi-infinite solid bounded by a plane surface and (2) along a straight wire of circular cross-section. Ch. V deals with the problem of diffraction and contains an interesting exposition of the multiform solutions of the wave-equation.

Ch. VI and the subsequent chapters deal respectively with transformations of co-ordinates appropriate for the treatment of problems connected with a surface of revolution, homogeneous solutions of the wave-equation, electro-magnetic fields with moving singularities, and miscellaneous theories.

The reader will realize that the book is not a complete treatise on the subject, although it is a great improvement on the late Prof.

Pockels' book, entitled "Ueber die partielle Differentialgleichung $\nabla^2 u + k^2 u = o$" (Teubner, 1891).

Dr. Bateman has succeeded in putting a great deal of information in a short volume. Almost every page contains detailed references to numerous original papers. To the advanced students of Applied Mathematics in English-knowing countries, the book should prove to be most useful.

CHANDI PRASAD.

On the invariant position of interferences in grating spectra.

BY

C. K. VENKATA ROW.

(*Communicated by Mr. C. V. Raman.*)

In the Philosophical Magazine for July 1910, Prof. C. Barus and Mr. M. Barus have described and explained the various systems of interference fringes crossing the spectrum that are seen when an Ives replica mounted between two plates of glass is used as a reflexion grating. They distinguish, in particular, between three different systems of interference bands and show that these may form the basis of a method of interferometry, if obtained with the aid of a separate movable reflector placed behind the grating. This particular aspect has been very fully worked out by Prof. C. Barus in an elaborate series of investigations.* But on a careful examination of Prof. Barus's papers, it appears that he has not observed a very remarkable property that I have found one of these interferences to possess. When a replica grating backed by a parallel reflecting surface separated from it by an air-space, is used as a reflexion grating, the strong vertical fringes which appear throughout the spectrum of the first order and which are identical with what Prof. Barus distinguishes as the total interferences, have been found to remain perfectly steady in space as the grating-mirror system is rotated. This rotation of the grating, keeping the collimator fixed, moves the spectrum only, but does not at all disturb the angular positions of the interferences in space, which, on the other hand, remain quite stationary. This remarkable property follows at once from the theory

* American Journal of Science, Nos. 177, 182, 194, 217, 222, 229, and 245.

Physical Review, Nov. 1910, Jan. 1916 and June 1916.

Philosophical Magazine, July 1910, April 1911, July 1911, June 1912 and Dec. 1912.

Science, March 1910, July 1910, Jan. 1911 and Dec. 1915.

Carnegie Publication, No. 149 (1911) on "The Production of Elliptic Interferences in relation to Interferometry."

of the bands. Let AB (Fig. 1) be the grating film and CD the reflecting surface. The ray PO is diffracted at the grating element O, along

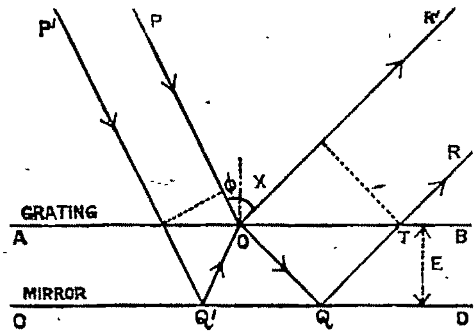

Fig. 1.

O Q and then reflected out at Q along Q R; and the ray P' Q' is reflected first at Q' along Q' O, meeting the same grating element O, where it is diffracted out along O R' parallel to Q R. The emergent parallel rays O R' and Q R interfere. The path-difference between them

$$=2t\,(\cos\chi-\cos\phi)$$

and therefore the condition for interference is

$$2t\,(\cos\chi-\cos\phi)=(2\kappa+1)\,\frac{\lambda}{2} \quad \ldots \quad \ldots \quad (1)$$

when k is any integer. (This equation has been obtained by Prof. Barus).

Now, if the direction defined by χ be a principal direction for light of wave-length λ we have

$$d\,(\sin\phi-\sin\chi)=n\lambda \quad \ldots \quad \ldots \quad (2)$$

where d is the grating interval.

If the light of wave-length λ is to interfere, equations (1) and (2) must be simultaneously satisfied. Therefore, dividing the corresponding sides of the equations, we have

$$\frac{2t\,(\cos\chi-\cos\phi)}{d\,(\sin\phi-\sin\chi)}=\frac{2\kappa+1}{2n}$$

$$\text{or}\quad \tan\frac{\phi+\chi}{2}=\frac{(2\kappa+1)d}{4nt} \quad \ldots \quad \ldots \quad (3)$$

This equation gives the directions in which the dark fringes are observed. By giving successive integral values to κ, we obtain the corresponding values of $(\phi+\chi)$ which give the angular positions of the fringes in relation to the direction of the incident light. These angular positions are determined once for all by equation (3), independent of the position of the grating. This fact explains the absolute fixity of the fringes which, again, is a beautiful geometrical confirmation of the theory of the bands.

In order to study the absolute fixity of the fringes and also to make a thorough quantitative verification of equation (3), the following arrangement was adopted: one of Thorp's copies of a Rowland grating, mounted on a glass slab for support, was placed with its film facing a thin glass plate whose front surface had been silvered. Two lengths of the same wire served as distance-pieces so that the grating and the silvered surface had a parallel film of air between them. With this arrangement, the system of bands was found to be perfectly fixed. A photograph of the bands when the air-space was 0 0740 cms. thick, is reproduced in the accompanying Plate. The source of light was the crater of an arc between carbon poles.

In order to test equation (3) over a fairly wide range, any dark fringe conveniently chosen, was made the starting point and the value of $(\phi+\chi)$ corresponding to every ten or twenty fringes was determined, and the thickness of the air film was deduced from the following slightly altered form of equation (3):

$$\tan\frac{\theta_q}{2} - \tan\frac{\theta_0}{2} = \frac{Qd}{2tn}$$

where θ_0 is the value of $(\phi+\chi)$ for the initial fringe and θ_q that of the Qth fringe from the starting point. This calculated value was compared with that given by the diameter of the interposed wire.

The following table contains an abstract of observations made to test the theory.

Experiment 1.—Diameter of interposed wire = 0·0740 Cms. Direct reading of spectrometer telescope = 199° 50′

Denote $\tan\frac{\theta_q}{2} - \tan\frac{\theta_0}{2}$ by I and the telescope reading by T.

TABLE I.

Spectrum of the first order. $\frac{\theta_0}{2} = 27^\circ\, 13'$

Q	T		$\frac{\theta}{2}$		Tan $\frac{\theta}{2}$	I	I/Q
0	74°	16′	27°	13′	·5143	0	......
10	75°	19′	27°	45′	·5261	·0118	·001180
20	76°	22′	28°	16′	·5377	·0234	·001170
40	78	28	29	19	·5616	·0473	·001182
60	80	31	30	21	·5856	·0713	·001188
80	82	32	31	21	·6092	·0949	·001186
100	84	28	32	19	·6326	·1183	·001183
120	86	24	33	17	·6565	·1422	·001185
140	88	17	34	14	·6805	·1662	·001188
160	90	6	35	8	·7037	·1894	·001184
180	91	54	36	2	·7274	·2131	·001184
200	93	40	36	55	·7513	·2370	·001185
						Mean:	·001185

In this case $d = \frac{2 \cdot 54}{14508}$ and from Table I we get I/Q = 0·001185.

Hence t (calculated) = 0·0738 cms for the thickness of the air-film. The measured diameter of the interposed wire was 0·0740 cms. The agreement is very satisfactory.

A second experiment with the same grating but with wires of different gauge for distance-pieces gave 0·1803 cms. as the thickness of the air-film against 0·1802 cms for the measured diameter of the wires. The agreement is again very satisfactory.

From Table I, we see that measurements have been made on 200 fringes; but it must be noticed that all the 200 fringes are not simultaneously in view on the spectrum; but the property of absolute fixity of the fringes in space irrespective of the position of the grating, enables us to bring the illumination to bear at any angle with the collimator, so that measurements can be made over a very large number of bands. This property furnishes also a very good device for photographing the fringes, since the violet and ultra-violet regions of the spectrum, which are largely responsible for the photographic action on ordinary plates, can be made to bear upon different parts of the photographic plate without disturbing the band system.

It is interesting to study how the fixity of the fringes is modified by the substitution of a dispersive medium like glass for the air-space. With this alteration, the experiment was carried out and the system was found still to be very nearly steady as the grating was rotated. But there was a slight motion of the fringes, which exhibited certain peculiar features. The movements of the fringes were in opposite directions at the ends of the spectrum, while in the region about the F—line there was no motion at all; the movement at the red end was more rapid than at the violet. Moreover, it was found that the bands at the red end move in the direction of the spectrum, when the spectrum in question is that nearer the collimator; but in the other spectrum away from the collimator, the fringes at the red end move in a direction contrary to that of the spectrum.

The effects described in the preceding para. may be explained by a consideration of the theory of the bands for a dispersive medium.

Let A B C D be the glass slab, (Fig. 2), the face A B carrying the grating-film. Let a parallel pencil P Q P′ Q′ of wave-length λ be incident on the grating. Let the ray P Q be incident on the line *Q* of the grating. It is there diffracted into the slab along Q R, then regularly reflected at the face C D along R S and finally refracted out into the air along the direction S T. The ray P′ Q′ refracted into the slab along Q′ R′, then regularly reflected at the face C D along R′ Q and finally diffracted out into air by the same grating line Q, along a direction Q T′ parallel to the direction S T. These two rays Q T′ and S T interfere.

Let us adopt the following notation in order to work out the path difference.

μ = refractive index of slab.

d = grating interval.

t = thickness of slab.

ϕ = angle of incidence on the grating.

χ = angle of diffraction into air.

Let ϕ' and χ' be defined by the equations

$$\text{Sin } \phi = \mu \sin \phi'.$$

$$\text{Sin } \chi = \mu \sin \chi'.$$

Now the path difference (see fig.)

$$= (\overline{EQ} + \mu \overline{QR} + \mu \overline{RS}) - (\mu \overline{Q'R'} + \mu \overline{R'Q} + QF)$$

$$= (EQ + 2 \mu \overline{QR}) - [2 \mu \overline{Q'R'} + QF)$$

But $\quad EQ = 2\, t \tan \phi' \sin \phi.$

$$QR = 2\, t \sec \chi'.$$

$$Q'R' = 2\, t \sec \phi'.$$

$$QF = 2\, t \tan \chi' \sin \chi.$$

Substituting, the path difference

$$= (2\, t \tan \phi' \sin \phi + 2\, \mu\, t \sec \chi').$$

$$- (2\, \mu\, t \sec \phi' + 2\, t \tan \chi' \sin \chi).$$

$$= (2\, t \tan \phi' \sin \phi - 2\, \mu\, t \sec \phi').$$

$$- (2\, t \tan \chi' \sin \chi - 2\, \mu\, t \sec \chi').$$

$$= \mu\, t \left(\frac{\sin \phi' \sin \phi}{\mu \cos \phi'} - \frac{1}{\cos \phi'}\right) - 2\, \mu\, t \left(\frac{\sin \chi' \sin \chi}{\mu \cos \chi'} - \frac{1}{\cos \chi'}\right)$$

$$= 2\, \mu\, t\, (\cos \chi' - \cos \phi').$$

The equation giving the angular position of the interferences is thus :—

$$2\, \mu\, t\, (\cos \chi' - \cos \phi') = \frac{2\, k + 1}{2} \lambda \,\ldots\ldots\ldots 1(a).$$

But the grating equation is

$$d\, (\sin \phi - \sin \chi) = n\, \lambda.$$

$$\text{or } \mu\, d\, (\sin \phi' - \sin \chi') = n\, \lambda \,\ldots\ldots\ldots\ldots\, 2(a).$$

Dividing (1a) by (2 a) we have.

$$\tan \frac{\phi' + \chi'}{2} = \frac{(2k+1)\,d}{4\,t\,n}$$

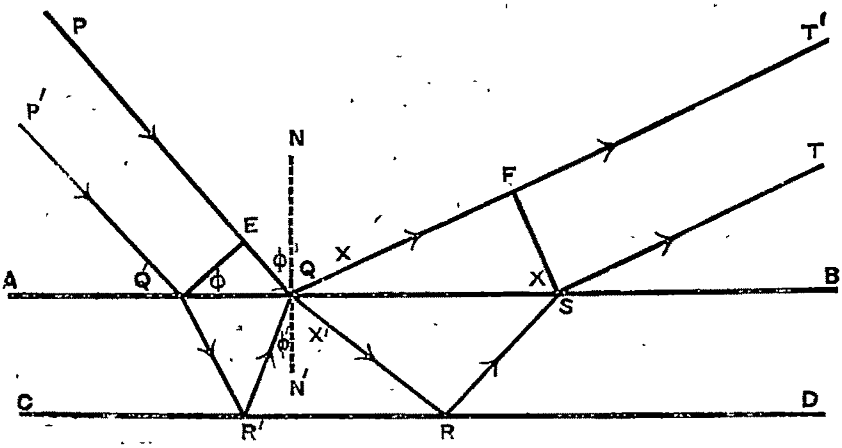

Fig. 2.

Thus we see that $(\phi' + \chi')$ is invariant for any particular fringe.

In order to study the motion of the fringes as the spectrum is moved, let us fix our attention upon one of the fringes (say that defined by $\phi' + \chi' = a$; where a is a constant angle), and study the variation of the angle $(\phi + \chi)$ corresponding to the fringe, as different wave-lengths pass it. To do this, let us write the equations connecting $\phi, \chi, \phi', \chi', \mu$ and λ. They are:

$$\text{Sin } \phi = \mu \sin \phi' \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4).$$

$$\text{Sin } \chi = \mu \sin \chi' \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5).$$

$$\text{Sin } \phi - \sin \chi = \frac{\lambda}{d} = b \text{ (say for brevity)} \quad \ldots \quad \ldots \quad (6).$$

and we have for the fringe in question

$$\phi' + \chi' = a \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Therefore $\sin \chi = \mu \sin (a - \phi')$.

$$= \sin a \sqrt{\mu^2 - \sin^2 \phi} - \cos a \sin \phi.$$

But from (6) we have

$$\text{Sin } \chi = \sin \phi - b.$$

Therefore

$$\text{Sin}\,\phi - b = \sin a\,\sqrt{\mu^2 - \sin^2\phi} - \cos a \sin\phi$$

or $\{(1+\cos a)\sin\phi - b\}^2 = \sin^2 a\,(\mu^2 - \sin^2\phi)$

or $\{(1+\cos a)^2 + \sin^2 a\}\sin^2\phi - 2b\,(1+\cos a)\sin\phi$

$$+ (b^2 - \mu^2 \sin^2 a) = 0$$

or $\sin^2\phi - b\sin\phi + \dfrac{b^2 - \mu^2\sin^2 a}{2\,(1+\cos a)} = 0$

or solving for $\sin\phi$

$$2\sin\phi = b \pm \sqrt{b^2 - \frac{2\,(b^2 - \mu^2\sin^2 a)}{1+\cos a}}$$

Substituting this value of $\sin\phi$ in equation (6) we have for $\sin\chi$

$$2\sin\chi = -b \pm \sqrt{b^2 - \frac{2\,(b^2 - \mu^2\sin^2 a)}{1+\cos a}}$$

Now denote for brevity the quantity under the radical sign by a.

Then we have

$$\text{Sin}\,\phi + \sin\chi = \pm a \qquad \ldots \qquad (7)$$

$$\text{Sin}\,\phi - \sin\chi = b \qquad \ldots \qquad (8)$$

Equation (7) can be expressed as

$$\tan\frac{\phi+\chi}{2}\sin(\phi+\chi)\left\{1+\cos(\phi-\chi)\right\} = a^2$$

$$\text{or}\ \cos^2(\phi-\chi) = \left\{a^2\cot\frac{\phi+\chi}{2}\operatorname{cosec}(\phi+\chi) - 1\right\}^2$$

and from (7) and (8) we have

$$\text{Sin}^2(\phi-\chi)) = a^2 b^2 \operatorname{cosec}^2(\phi+\chi)$$

Adding the two and simplifying

$$\left(a^2\cot^2\frac{\phi+\chi}{2} + b^2\right)\operatorname{cosec}(\phi+\chi) = 2\cot\frac{\phi+\chi}{2}$$

$$\text{or}\quad a^2 + b^2\tan^2\frac{\phi+\chi}{2} = 4\sin^2\frac{\phi+\chi}{2} \qquad \ldots \qquad \ldots \quad (9)$$

But now

$$a^2 = b^2 - \frac{2\,(b^2 - \mu^2 \sin^2 a)}{1+\cos a}$$

$$= \frac{2\mu^2 \sin^2 a}{1+\cos a} - \frac{1-\cos a}{1+\cos a} b^2$$

$$= 4\mu^2 \sin^2 \frac{a}{2} - b^2 \tan^2 \frac{a}{2}$$

Therefore, equation (9) becomes after restoring λ^2/d^2 for b^2.

$$\frac{\lambda^2}{d^2}\left(\tan^2 \frac{\phi+\chi}{2} - \tan^2 \frac{a}{2} \right) + 4\mu^2 \sin^2 \frac{a}{2}$$

$$= 4 \sin^2 \frac{\phi+\chi}{2} \qquad \ldots \qquad \ldots \qquad (10)$$

This equation furnishes us the values of the angle $(\phi+\chi)$ which the fringe in question makes with the axis of the collimator, as light of different wave-lengths passes it. We can get a clear idea of the small motion of the fringe, if we work out from equation (10) a numerical example. Let us choose the fringe defined by $a=44°$ and let the grating contain 6000 lines to the centimetre. Then assuming the following values of μ, we arrive at the values of $\frac{\phi+\chi}{2}$ tabulated in Table II: (these values were obtained from equation (10) by the method of successive approximations, since the term involving λ^2 is comparatively small; it is found that two successive approximations suffice).

$a=44°$

TABLE II.

Line.	λ (A-U).	λ^2/d^2.	μ	$\frac{\phi+\chi}{2}$
A	7594	0·2076	1·528	36°4′
B	6870	0·1700	1·530	35°54′
C	6563	0·1551	1·531	35°50′
D	5893	0·1250	1·534	35°45′
E	5270	0·1000	1·537	35°42′
F	4861	0·0851	1·540	35°41′
G	4308	0·0668	1·546	35°45

From this table we see that the angle $(\phi+\chi)$ is very nearly constant, and this accounts for the approximate fixity of the frings. In order to show the nature of the small motion, the relation between the corresponding values of $\frac{\phi+\chi}{2}$ and λ is exibited in the curve (Fig. 3.)

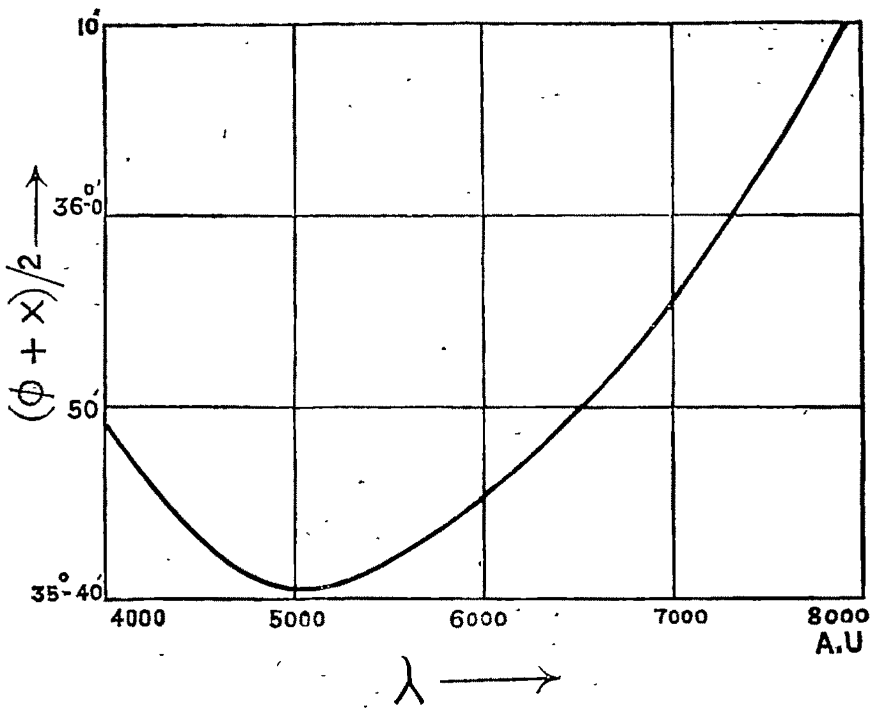

Fig. 3.

From the curve we see that, as the wavelength of the light passing any particular fringe gradually diminishes, the angle $(\phi+\chi)$ which it makes with the collimator, diminishes also, more rapidly at first, till it attains a minimum in the blue-green region about $\lambda=5000$ A. U., and thereafter, the angle goes on increasing in the violet and the ultra-violet. This indicates clearly the existence of a minimum of motion in the blue-green region with a comparatively greater motion in the red than in the violet, and explains why the motion should be in opposite directions at the ends of the spectrum as observed before.

It may, at first sight, seem that the dispersion of the glass has very little to do with the small motion of the fringes and that a

constant value of μ can explain the phenomenon as well; but a little consideration of equation (10) shows that in that case, $(\phi+\chi)$ is a minimum only when it is equal to a which can never be the case. It is to the dispersion that we must ascribe the minimum indicated by the curve (Fig. 3).

As regards the magnitude of the motion, we see from Table II, that the variation of $(\phi+\chi)$ is about $26'$, as the spectrum moves past the fringe from the B-line to the F-line. It can easily be calculated that with a thickness of 0·5 cms. of crown glass between the grating and the reflecting surface in the numerical example we have worked out, the angular width between two consecutive fringes, when the angle $(\phi+\chi)=71°$, is (about) $1'\ 45''$, so that for the motion of the spectrum referred to above, there must be a movement of about fifteen fringes past the cross-wire of the telescope. This result was tested experimentally under the same circumstances as those stated above. It was found that twelve fringes moved past the cross-wire as the spectrum moved past the fringe from the B-line to the F-line. The agreement is good, considering the nature of the calculations and the uncertainty due to the lack of knowledge of the exact dispersion of the glass used; the values of μ introduced into the calculation are those given for crown glass in physical tables. The fringes observed in the above experiment with 0·5 cms. of crown glass, were photographed; the photograph (not reproduced) gives a very good idea of the smallness of the motion of the fringes, when it is seen how small a space twelve of the closely packed fringes, taken consecutively, occupy on a plate which covers nearly the whole of the visible spectrum.

Summary and Conclusion.

It has been shown that the strong system of fringes in the spectrum formed by a replica grating backed by a parallel reflecting surface with air-space between, possesses the extraordinary property of absolute fixity in space irrespective of the angular position of the grating with reference to the incident light. This system of fringes is unique in this respect and it is this characteristic feature that distinguishes it from the other systems of bands observed by Prof. Barus. We have also observed that the fringes do nearly but not quite retain this characteristic when the air space is replaced by a dispersive medium. But in this case, the interesting feature is that the fringes move slightly in opposite directions at the opposite ends of the spectrum when the grating-mirror is rotated, while in the blue-green region they are quite steady.

The experiments and observations recorded in this paper were carried out at the Physical Laboratory of the Presidency College, Madras. I wish to express my gratitude to Prof. R. L. Jones, M.A., F.R.A.S. and also to Prof. C. V. Raman, M.A., of Calcutta, for the encouragement and advice received by me in the course of the work.

PRESIDENCY COLLEGE, MADRAS,
The 15th January, 1917.

The construction of certain peculiar rectifiable curves.

By

Lakshmi Narayan.

As is well-known, according to the ordinary text-books on the Differential Calculus, at every point of a curve $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$. The object of the present paper is to construct a family of rectifiable curves each of which possesses the property that, at certain points, although $\frac{dx}{ds}$ and $\frac{dy}{ds}$ exist, $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2$ is *not* equal to unity, where, of course, x and y are the cartesian co-ordinates of a point and s is the length of the arc measured from a fixed point on the curve. It is believed that this family of curves has not been given by any previous writer.*

In Art 1, I begin by defining one such curve and in Art 2, I prove that this curve is rectifiable. In Art 3, the general values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are found, t being the variable parameter in terms of which x and y are given, and in Art 4, the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at the special point $t=w$, have been found. In Arts. 5 and 6, it is proved that $+\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is *not* equal to $\frac{ds}{dt}$ at the point $t=w$, while $+\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ *is* equal to $\frac{ds}{dt}$ at the other points of the interval for which the curve is defined. In Art. 7 it is indicated that the curve defined in Art. 1 *gives* a *family* of curves if the constants involved in the definition be considered as variable parameters. In Art 8, I show how to construct a family of rectifiable

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* An isolated example of a curve for which $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 \neq 1$, for certain points, has been given by Dr. Julius Perl. (See *Jahresbericht der deutschen Mathematiker—Vereinigung*, Vol. 18, 1909, pp. 399-401).

curves of greater generality, each curve possessing the property that $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2$ is different from unity *not at a single point* but at a *finite* or *enumerably infinite* number of points.

My thanks are due to Dr. Ganesh Prasad at whose suggestion I took up the investigation.

$$\left.\begin{aligned} 1. \quad \text{Let } x &= \int_a^t \left\{ F_1(t) + F_2(t) + F_3(t) \right\} dt \quad . \quad . \quad . \\ \text{and } y &= \int_a^t 2\,Ak\,\phi(t)\,dt \quad . \quad . \quad . \end{aligned}\right\} \ldots (1)$$

$$(a < t \leq b);$$

and $x=o$, $y=o$ for $t=a$;

where $F_1(t) = \frac{k^2(n+1)^2}{2}(t-w)^{2n} + \frac{k^2n^2c^2}{2} - A^2$

for all values of t in (a, b);

$$F_2(t) = -k^2\left[\frac{(n+1)^2}{2}(t-w)^{2n}\cos\frac{2c}{(t-w)^n} + \tfrac{3}{2}nc(n+1)(t-w)^n \sin\frac{2c}{(t-w)^n}\right]$$

for $t \neq w$

and $F_2(t) = o$ for $t=w$;

$$F_3(t) = -\tfrac{1}{2}cnk^2\left[(n+1)(t-w)^n\sin\frac{2c}{(t-w)^n} - 2nc\cos\frac{2c}{(t-w)^n}\right]$$

for $t \neq w$

and $F_3(t) = p$, any finite real number, for $t=w$;

$$\phi(t) = (n+1)(t-w)^n \sin\frac{c}{(t-w)^n} - cn\cos\frac{c}{(t-w)^n}, \text{ for } t \neq w$$

and $\phi(t) = q$, any finite real number, for $t=w$;

k and c are finite real numbers; n is a positive number;

A is a constant different from zero;

w is any number lying in the interval (a, b).

2. In the above, $F_1(t)$ and $F_2(t)$ are finite and continuous functions of t for all values of t in the interval (a, b), and $F_3(t)$ and $\phi(t)$ are finite, and everywhere continuous in (a, b) excepting the point $t=w$ where they have discontinuities of the second kind. Hence the integrals defining x and y exist, and x and y are continuous functions of t and of limited total fluctuation in (a, b).* Also, since for a function with limited total fluctuation, and without any point at which there is an external saltus, the total fluctuation and the total variation are the same,§ it follows that x and y are functions of t with bounded variation. Now the necessary and sufficient condition † that a curve defined by the equations $x=\psi(t)$, $y=\chi(t)$, may be rectifiable, is that $\psi(t)$ and $\chi(t)$ should be functions of bounded variation of t. Hence the curve defined by equations (1) is a rectifiable curve.

3. Since $F_1(t)+F_2(t)+F_3(t)$ and $2Ak\,\phi(t)$ are continuous at any point other than w in the interval (a, b), it is evident that

$$\frac{dx}{dt} = F_1(t)+F_2(t)+F_3(t)$$

$$\text{and} \quad \frac{dy}{dt} = 2Ak\,\phi(t)$$

for all values of t in (a, b)** *except* w.

Now, for $t \neq w$, $F_1(t)+F_2(t)+F_3(t)$

$$= \left\{ \frac{k^2(n+1)^2}{2}(t-w)^{2n} + \frac{k^2n^2c^2}{2} - A^2 \right\}$$

$$-k^2 \left\{ \frac{(n+1)^2}{2}(t-w)^{2n} \cos\frac{2c}{(t-w)^n} + \tfrac{3}{4}nc\,(n+1)\,(t-w)^n \right.$$

$$\left. \sin\frac{2c}{(t-w)^n} \right\} - \tfrac{1}{4}nck^2 \left\{ (n+1)(t-w)^n \sin\frac{2c}{(t-w)^n} \right.$$

$$\left. \angle\, 2nc \cos\frac{2c}{(t-w)^n} \right\} = -A^2 + \frac{k^2(n+1)^2}{2}(t-w)^{2n}$$

$$\left\{ 1-\cos\frac{2c}{(t-w)^n} \right\} + \frac{k^2n^2c^2}{2} \left\{ 1+\cos\frac{2c}{(t-w)^n} \right\}$$

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* See Hobson's *Theory of functions of a real variable*, Art. 258.

§ Hobson, *loc. cit.*, Art 196.

† See Lebesgue's "*Lecons sur l'integration*" page 60 where Lebesgue gives the necessary and sufficient conditions of rectifiability for the general case of curves in space. But the proof given applies equally to plane curves.

** See Hobson, *loc. cit.* Art. 260.

$$-n\,(n+1)\;ck^2\;(t-w)^n\;\sin\frac{2c}{(t-w)^n}$$

$$=-\,A^2+k^2\,(n+1)^2\,(t-w)^{2n}\,\sin^2\frac{c}{(t-w)^n}$$

$$+k^2n^2c^2\,\cos^2\frac{c}{(t-w)^n}\;-\;2n\,(n+1)\;ck^2\;(t-w)^n\;.$$

$$\sin\frac{c}{(t-w)^n}\;\cos\frac{c}{(t-w)^n}\;=k^2\,\phi^2-A^2$$

Hence for $t\neq w$,

$$\left.\begin{aligned}\frac{dx}{dt}&=k^2\phi^2-A^2 \quad \ldots \quad \ldots\\ \frac{dy}{dt}&=2\,A\,k\phi\,(t) \quad \ldots, \quad \ldots,\end{aligned}\right\}\quad\ldots\quad(2)$$

From the above it is clear that

$$x=\int_a^t\,(k^2\phi^2-A^2)\;dt\;\S\qquad\ldots\qquad\ldots\qquad(3)$$

since $F_1+F_2+F_3=k^2\phi^2-A^2$ at all points in $(a,\,b)$ except w, and the value of an integral is unaffected by changing the value of the integrand at a particular point.

4. Let us now investigate the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ for $t=w$.

We have $x=\int_a^t\,(F_1+F_2+F_3)\;dt=\int_a^t F_1\,dt+\int_a^t F_2\,dt+\int_a^t F_3\,dt.$

Since F_1 and F_2 are continuous at the point $t=w$, the contributions of $\int F_1\,dt$ and $\int F_2\,dt$ towards the value of $\frac{dx}{dt}$ at $t=w$, are $\underset{t=w}{\mathrm{L}}\,F_1$ and $\underset{t=w}{\mathrm{L}}\,F_2$ *i.e.* $\frac{k^2n^2c^2}{2}-A^2$ and zero respectively.

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§ x has been defined by equation (1) rather than by (3) to avoid a certain obscurity of explanation in the deduction of the value of $\frac{dx}{dt}$ for $t=w$.

Again $\displaystyle\int_a^t F_3\, dt = -\tfrac{1}{4}nck^2 \int_a^t \Big\{ (n+1)\ (t-w)^n \sin\frac{2c}{(t-w)^n}$

$$- 2nc \cos\frac{2c}{(t-w)^n} \Big\} dt.$$

$= -\tfrac{1}{4}\, nck^2.\ (t-w)^{n+1} \sin\dfrac{2c}{(t-w)^n} + \alpha$, where α is a known constant.

Also, since $\displaystyle\int_a^t F_3\, dt$ is a continuous function* of t,

the value of $\displaystyle\int_a^t F_3\, dt$ for $t=w$ is $\displaystyle\mathop{L}_{t=w} \int_a^t F_3\, dt.$

$$= \mathop{L}_{t=w} -\tfrac{1}{4}nck^2(t-w)^{n+1} \sin\frac{2c}{(t-w)^n} + \alpha$$

$$= \alpha.$$

Hence the part contributed by $\displaystyle\int_a^t F_3\, dt$ to the value of $\dfrac{dx}{dt}$ at the point $t=w$, is

$$\mathop{L}_{t=w} \frac{1}{(t-w)} \Big\{ \int_a^t F_3\, dt - \alpha \Big\}$$

$$= \mathop{L}_{t=w} \frac{1}{(t-w)} (-\tfrac{1}{4}nck^2)\ (t-w)^{n+1} \sin\frac{2c}{(t-w)^n}$$

$$= 0.$$

Therefore, when $t=w$, $\dfrac{dx}{dt}$

$$= \frac{k^2n^2c^2}{2} - A^2. \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

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* See Hobson, *loc. cit.*, Art. 258.

To find the value of $\frac{dy}{dt}$ for $t=w$, we have

$$y=2\mathrm{A}\,k\int_a^t \phi\,(t)\,dt=2\mathrm{A}k\int_a^t\Big[(n+1)(t-w)^n\,sin\,\frac{c}{(t-w)^n}$$

$$-\,cn\,cos\,\frac{c}{(t-w)^n}\Big]\,dt$$

$$=2\mathrm{A}k\,\Big[(t-w)^{n+1}\,sin\,\frac{c}{(t-w)^n}-(a-w)^{n+1}\,sin\,\frac{c}{(a-w)^n}\Big].$$

Since y is a continuous function of t, the value of y for $t=w$ is

$$\underset{t=w}{\mathrm{L}}\;2\mathrm{A}k\;\Big\{(t-w)^{n+1}\,sin\,\frac{c}{(t-w)^n}-(a-w)^{n+1}\,sin\,\frac{c}{(t-w)^n}\Big\}$$

$$=-\,2\mathrm{A}k\,(a-w)^{n+1}\,sin\,\frac{c}{(a-w)^n}\qquad\ldots\qquad\ldots\qquad(5)$$

$$(a<w\leq b)$$

[If w happens to be equal to a, the value of y for $t=w$ is the same as the value of y when $t=a$, which has already been defined as zero,]

Hence when $t=w$, the value of $\frac{dy}{dt}$ is

$$\underset{t=w}{\mathrm{L}}\;\frac{1}{(t-w)}\;\Big[2\,\mathrm{A}k\;\Big\{(t-w)^{n+1}\,sin\,\frac{c}{(t-w)^n}-(a-w)^{n+1}$$

$$sin\,\frac{c}{(a-w)^n}\Big\}+2\mathrm{A}k\,(a-w)^{n+1}\,sin\,\frac{c}{(a-w)^n}\Big]$$

$$=\underset{t=w}{\mathrm{L}}\;2\mathrm{A}k\,(t-w)^n\,sin\,\frac{c}{(t-w)^n}$$

$$=0\qquad\ldots\qquad\ldots\qquad\ldots\qquad(6)$$

5. The values of $\frac{dv}{dt}$ and $\frac{dy}{dt}$ at the point $t=w$ are given by the equations (4) and (6) and their values at the other points of the interval $(a,\,b)$ are given by equations (2). We observe that $\frac{dv}{dt}$ and $\frac{dy}{dt}$ are

finite, and everywhere continuous, in (a, b) except at w. The same is true of their squares and the square root of the sum of their squares. Hence $+\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is integrable in any interval lying in (a, b). Consequently, the length of the arc of the curve defined by equations (1) is given by the equation

$$s = \int_a^t + \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt^*$$

which is seen to be equivalent to

$$s = \int_a^t + \sqrt{\{k^2\phi^2 - A^2\}^2 + 4A^2k^2\phi^2}\, dt$$

on substituting the general values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ from equations (2), for the integral is unaffected by the value of $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ at the particular point w.

$$\text{Therefore } s = \int_a^t \{k^2\phi^2 + A^2\}\, dt \quad \ldots \quad \ldots \quad (7)$$

Since the expression $k^2\phi^2 + A^2$ is continuous for all points other than w in (a, b), $\frac{ds}{dt} = k^2\phi^2 + A^2$ when $t \neq w$ ... ... (8)

Comparing equations (3) and (7), we see that s differs from x only in having $+A^2$ instead of $-A^2$ in the expression under the integral sign. Hence when $t = w$, $\frac{ds}{dt}$ will be equal to $\frac{k^2n^2c^2}{2} + A^2$: because from equation (4) $\frac{dx}{dt} = \frac{k^2n^2c^2}{2} - A^2$ when $t = w$.

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* See Lebesgue, *loc. cit.* p. 61.

6. Now, at all points other than w in (a, b),

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = k^2\phi^2+A^2 = \left(\frac{ds}{dt}\right)$$

from equations (2) and (8).

Therefore multiplying by $\frac{dt}{ds}$, we have

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \text{ for } t \neq w,$$

the constants k, n, c, A being so chosen that $k^2\phi^2+A^2$ is neither zero nor infinity.

But when $t=w$; $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{k^2n^2c^2}{2} - A^2$

and $\frac{ds}{dt} = \frac{k^2n^2c^2}{2} + A^2$

Hence $+\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$ is *not* equal to $\frac{ds}{dt}$ when $t = w$,

because A is not zero,

and therefore $\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2$ is not equal to unity when $t=w$.

Here we have, of course, assumed that

(i) If $\frac{ds}{dt}$ exists as a finite definite number different from zero,

then $\frac{dt}{ds}$ exists and $\frac{ds}{dt} \times \frac{dt}{ds} = 1$.

(ii) If $\frac{dx}{dt}$ and $\frac{dt}{ds}$ exist, as finite numbers, then $\frac{dx}{ds}$ exists and

$$\frac{dx}{dt} \times \frac{dt}{ds} = \frac{dx}{ds}.$$

7. In the equations (1) of the curve defined in Art. 1 the constants k, n, c, w, A occur.

If these are supposed to be variable parameters, we have a family of curves possessing the property discussed above, or we may connect these five constants by one, two, three or four relations among themselves, provided always that A shall not be zero. Let us suppose that n is a positive number depending upon w, and k and c are constants or functions of w, then giving to w all the real values between a and b, we have a family of curves, such that $\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2$ is different from unity at *each* point of the interval (a, b) for *some one or other* member of the family. Also, by properly choosing the constant A, we can make $\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2$ equal to any arbitrary number lying between $+1$ and zero.

8. Arts 1—6 suggest that the procedure, indicated below, gives rectifiable curves for each of which $\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2$ is different from unity at a finite or an enumerably infinite number of points $\{w_n\}$.

Let $f(t)$ be a function of t, which is finite, continuous and differentiable everywhere in the interval (a, b) and which vanishes for a finite or enumerably infinite number of points ($t=w_1$, $t=w_2,\ldots$) in (a, b). Also, let us suppose that $f'(t)$ is integrable in (a, b) and continuous and different from zero for all values of t for which $f(t)$ vanishes. Then a curve of the required type is defined in the following way:—

$$x=\int_a^t \{F_1(t)+F_2(t)+F_3(t)\}\, dt,$$

$$y=2Ak\int_a^t \phi(t)\, dt;$$

where $F_1(t)=\frac{1}{2}\, k^2c^2n^2f'^2-A^2$ for all values of t in $(a,\ b)$;

$$F_2(t)=k^2\left\{\frac{(n+1)}{2}\, f^{2n}f'^2\left(1-\cos\frac{2c}{f^n}\right)-\frac{3c}{4}\, n(n+1)\, f'^2f^n\sin\frac{2c}{f^n}\right\}$$

for all values of t for which f does not vanish,

and $F_2(t) = 0$ for those values of t for which f vanishes;

$$F_3(t) = -\frac{k^2 cnf'}{4}\left\{ (n+1)\, f' f^n \sin\frac{2c}{f^n} - 2ncf' \cos\frac{2c}{f^n} \right\}$$

for all values of t for which f does not vanish,

and $F_3(t) = \rho_1,\ \rho_2,\ \rho_3 \ldots$ corresponding to the values $t = w_1,\ t = w_2,\ t = w_3 \ldots$ for which f vanishes, $\rho_1,\ \rho_2,\ \rho_3 \ldots$ being any finite real numbers;

$A,\ n,\ k,\ c$ are numbers as in Art. 1;

$$\phi(t) = (n+1)\, f^n\, f' \sin\frac{c}{f^n} - cnf' \cos\frac{c}{f^n}$$

for those values of t for which f does not vanish,

and $\phi(t) = q_1,\ q_2,\ \ldots\ q_3 \ldots$

for $t = w_1,\ w_2,\ \ldots\ w_3 \ldots$ respectively,

$q_1,\ q_2,\ q_3 \ldots$ being any real finite numbers.

[*N.B.*—Thoughout this paper, the notion of integral adopted is that of Riemann.]

On the determination of a rough surface on which a moving particle may describe a prescribed path.

BY

NALINIMOHAN BASU.

The object of the present paper is to show how the solution of the following problem can be made to depend on the solution of an *ordinary linear differential equation*:

"To find the *rough* surface on which a moving particle may describe a prescribed curve."

A very simple case of this problem, *viz.*, that in which the surface is *smooth* and gravity the only external force, was studied by the distinguished Belgian mathematician Catalan.*

That the ordinary differential equation is generally not soluble by quadratures should not surprise us, because, as is well known, the motion of a particle on a rough surface has been shown to be determinable by quadratures in only a small number of cases.**

I should like to express my indebtedness to Dr. Ganesh Prasad, at whose suggestion I took up, and under whose guidance I carried on, the investigation of this problem.

1. Let us first consider the case when the external force is simply gravity.

Taking the axis of z vertically upwards and unity for the mass of the particle, the equations of motion are

$$\ddot{x} = l R - \mu R \frac{dx}{ds},$$

$$\ddot{y} = m R - \mu R \frac{dy}{ds},$$

$$\ddot{z} = n R - g - \mu R \frac{dz}{ds}$$

where l, m, n are the direction-cosines of the normal to the required surface, μ the co-efficient of friction and R the normal re-action.

* "Sur une probleme de mécanique" (*Journal de Mathématiques*, Series 1, tome 11).

** See p. 507 of Prof. Stäckel's article on dynamics in the "Encyclopädie der Mathematischen Wissenschaften" Vol. IV.

Eliminating R we have

$$\frac{\ddot{x}}{l - \mu \frac{dx}{ds}} = \frac{\ddot{y}}{m - \mu \frac{dy}{ds}} = \frac{\ddot{z} + g}{n - \mu \frac{dz}{ds}} = \frac{\frac{d}{dt}\left(\frac{v^2}{2} + gz\right)}{-\mu v} \quad (1)$$

Thus we have

$$\mu v \ddot{x} = \left(\mu \frac{dx}{ds} - l\right) \frac{d}{dt} \left(\tfrac{1}{2} v^2 + gz\right),$$

$$\mu v \ddot{y} = \left(\mu \frac{dy}{ds} - m\right) \frac{d}{dt} \left(\tfrac{1}{2} v^2 + gz\right).$$

Writing $\ddot{x} = \frac{d}{dt}\left(v \frac{dx}{ds}\right)$, $\ddot{y} = \frac{d}{dt}\left(v \frac{dy}{ds}\right)$, $\frac{d}{dt} = v \frac{d}{ds}$, and simplifying, we get

$$\mu v^2 \frac{d^2 x}{ds^2} + l v \frac{dv}{ds} + \left(l - \mu \frac{dx}{ds}\right) g \frac{dz}{ds} = 0,$$

$$\mu v^2 \frac{d^2 y}{ds^2} + m v \frac{dv}{ds} + \left(m - \mu \frac{dy}{ds}\right) g \frac{dz}{ds} = 0.$$

Hence we obtain

$$\frac{\mu v^2}{\mu\left(m \frac{dx}{ds} - l \frac{dy}{ds}\right)} = \frac{v \frac{dv}{ds}}{\left(l - \mu \frac{dx}{ds}\right) \frac{d^2 y}{ds^2} - \left(m - \mu \frac{dy}{ds}\right) \frac{d^2 x}{ds^2}}$$

$$= \frac{g \frac{dz}{ds}}{m \frac{d^2 x}{ds^2} - l \frac{d^2 y}{ds^2}} \quad \ldots \quad (2)$$

$$\therefore v^2 = \frac{g \frac{dz}{ds} \left(m \frac{dx}{ds} - l \frac{dy}{ds}\right)}{m \frac{d^2 x}{ds^2} - l \frac{d^2 y}{ds^2}} \quad \ldots \quad (3)$$

2. Now, let the equations of the given curve be

$$x = f_1(z), \; y = f_2(z),$$

f_1 and f_2 being two known functions.

Then, the equation of the required surface may be written in the form

$$\chi \equiv x - f_1(z) + \{ y - f_2(z) \} \phi(x, y, z) = 0 \; :$$

our problem is to determine the necessary form of ϕ.

Now

$$ds^2 = dx^2 + dy^2 + dz^2 = dz^2 \{ 1 + [f'_1(z)]^2 + [f'_2(z)]^2 \}$$

$$= \frac{dz^2}{\{\rho(z)\}^2},$$

where $\frac{1}{\{\rho(z)\}^2}$ stands for $1 + [f'_1(z)]^2 + [f'_2(z)]^2$.

$$\therefore \frac{dz}{ds} = \rho(z),$$

$$\frac{dx}{ds} = \rho(z) f'_1(z), \quad \frac{dy}{ds} = \rho(z) f_2'(z),$$

$$\frac{d^2x}{ds^2} = [\rho(z)]^2 f_1''(z) + \rho(z)\rho'(z) f_1'(z),$$

$$\frac{d^2y}{ds^2} = [\rho(z)]^2 f_2''(z) + \rho(z)\rho'(z) f_2'(z).$$

Also $\frac{\delta\chi}{\delta x} = 1 + \{y - f_2(z)\} \frac{\delta\phi}{\delta x}$

$= 1$ for all points on the curve.

Similarly $\frac{\delta\chi}{\delta y} = \mathrm{P}$ for all points on the curve,

where $\mathrm{P} = \phi\{f_1(z), f_2(z), z\}$;

$$\frac{\delta\chi}{\delta z} = -\{f_1'(z) + \mathrm{P} f_2'(z)\}.$$

$$\because \frac{l}{\frac{\delta\chi}{\delta x}} = \frac{m}{\frac{\delta\chi}{\delta y}} = \frac{n}{\frac{\delta\chi}{\delta z}} = \frac{1}{\Omega},$$

where $\Omega^2 = 1 + \mathrm{P}^2 + \{f'_1(z) + \mathrm{P} f'_2(z)\}^2$,

$$\therefore l = \frac{1}{\Omega}, \; m = \frac{\mathrm{P}}{\Omega} \text{ and } n = -\frac{\{f'_1(z) + \mathrm{P} f'_2(z)\}}{\Omega}$$

Hence from (3) we have

$$v^2 = \frac{g[\rho(z)]^2\{\mathrm{P} f_1'(z) - f_2'(z)\}}{\mathrm{P}[\{\rho(z)\}^2 f_1''(z) + \rho(z)\rho'(z) f_1'(z)] - [\{\rho(z)\}^2 f_2''(z) + \rho(z)\rho'(z) f_2'(z)]}$$

$= \theta'(z)$ say.

$$\therefore v\frac{dv}{ds} = \tfrac{1}{2}\theta'(z)\frac{dz}{ds} = \tfrac{1}{2}\rho(z)\theta'(z),$$

$\theta'(z)$ containing P and $\frac{dP}{dz}$.

If we substitute this value of $v\frac{dv}{ds}$ in the second of the equations (2) we get a differential equation in z from which P can be determined, although not necessarily by quadratures as the differential equation is generally of a complicated form and is insoluble by quadratures except in special cases.

3. Let the value of P thus determined be denoted by F (z). Then one of the forms in which $\phi(x, y, z)$ can be expressed is given by

$$\phi(x, y, z)=\mathrm{F}(z)+\{x-f_1(z)\}\psi_1(x, y, z)$$
$$+\{y-f_2(z)\}\psi_2(x, y, z),$$

where ψ_1 and ψ_2 are perfectly arbitrary functions.

Substituting this value of $\phi(x, y, z)$ in the equation of the surface we get a class of equations of the required surface.

4. Let us now consider the case in which the particle moves under *any conservative system of external forces.*

If the components of the external forces are given by $\mathrm{X}=\frac{\delta U}{\delta x}$, $\mathrm{Y}=\frac{\delta U}{\delta y}$, $Z=\frac{\delta U}{\delta z}$, the equations of motion will be

$$\ddot{x}=\mathrm{X}+l\mathrm{R}-\mu\mathrm{R}\frac{dx}{ds}, \text{ and two similar equations.}$$

Then proceeding exactly as in Art. 1. we derive the following two equations:—

$$\mu v^2\frac{d^2x}{ds^2}+lv\frac{dv}{ds}-\left(l-\mu\frac{dx}{ds}\right)\frac{dU}{ds}-\mu\frac{\delta U}{\delta x}=0,$$

$$\mu v^2\frac{d^2y}{ds^2}+mv\frac{dv}{ds}-\left(m-\mu\frac{dy}{ds}\right)\frac{dU}{ds}-\mu\frac{\delta U}{\delta y}=0,$$

whence we obtain

$$\frac{\mu v^2}{\mu\left[\left(l\frac{\delta U}{\delta y}-m\frac{\delta U}{\delta x}\right)-\left(l\frac{dy}{ds}-m\frac{dx}{ds}\right)\frac{dU}{ds}\right]}$$

$$=\frac{v\frac{dv}{ds}}{\frac{d^2y}{ds^2}\left[\left(l-\mu\frac{dx}{ds}\right)\frac{dU}{ds}+\mu\frac{\delta U}{\delta x}\right]-\frac{d^2x}{ds^2}\left[\left(m-\mu\frac{dy}{ds}\right)\frac{dU}{ds}+\mu\frac{\delta U}{\delta y}\right]}$$

$$=\frac{-1}{m\frac{d^2x}{ds^2}-l\frac{d^2y}{ds^2}}$$

But, on the curve, $\frac{dU}{ds}$, $\frac{\delta U}{\delta x}$, $\frac{\delta U}{\delta y}$ can be expressed in terms of z alone. Therefore, as in the previous case, we get a differential equation in z for P, from which P can be determined. If F (z) be the solution of this equation, the equation of the required surface, will be of the same form as in Art. 3.

5. *Illustrative examples.* (I) Let the prescribed path be the *circle*

$$x=z \tan a, \; x^2+y^2+z^2=1,$$

and let gravity be the only external force. Then, it is easily seen from Art. 2 that the ordinary differential equation for determining the required surface is

$$\frac{d}{dz}\left(\frac{P.\sin 2a}{1-2Pz\tan a}\right)=-3+\frac{\Omega\,\mu\sin 2a}{(1-2Pz\tan a)\sqrt{\cos^2 a-z^2}},$$

where

$$\Omega^2\cos^2 a=1-4Pz\tan a+4z^2P^2\sec^2 a+4P^2(\cos^2 a-z^2).$$

(II). Let the prescribed path be the *helix*

$$x=\cos kz, \; y=\sin kz,$$

under the action of gravity.

Then the ordinary differential equation for P is

$$\frac{d}{dz}\left(\frac{P\sin kz+\cos kz}{P\cos kz-\sin kz}\right)=2k\left(\frac{kc\,\mu\,\Omega}{P\cos kz-\sin kz}-1\right),$$

where $c^2=\frac{1}{1+k^2}$ and $\Omega^2=1+P^2+k^2$ (P $\cos kz-\sin kz)^2$.

If we put

$$\frac{P\sin kz+\cos kz}{\mathrm{P}\cos kz-\sin kz}=Q,$$

the above equation becomes

$$\frac{dQ}{dz}=-2k+2k^2c\,\mu\sqrt{Q^2+\frac{1}{c^2}}$$

which is soluble by quadratures.

On the non-stationary state of heat in an ellipsoid.

BY

BIBHUTIBHUSAN DATTA.

Introduction.

1. The first writer, who attempted, with some success, the problem of the determination of the non-stationary state of heat in an *ellipsoid with three unequal axes,* was E. Mathieu* who showed how the problem could be reduced to the solution of certain ordinary linear differential equations. But he found these equations to be so unmanageable that he contented himself with approximating to their solutions for the special case of an *ellipsoid of revolution.* Prof. C. Niven improved upon the results of Mathieu in certain respects in an interesting memoir†, entitled "On the conduction of heat in ellipsoids of revolution."

In the present paper, I propose (1) to obtain, and improve upon, the chief results of Prof. Niven by using an entirely different method, and (2) to show how this method can be applied to the case of the ellipsoid with three unequal axes to obtain similar results which are believed to be new.

It may be noted here that, in Art. 6, I point out a mistake in Prof. Niven's memoir.

I take this opportunity to express my indebtedness to Dr. Ganesh Prasad, under whose guidance I carried on the investigation the results of which are embodied in this paper.

Preliminary remarks, and definitions.

2. Let the initial temperature of the ellipsoid be $f(x, y, z)$ and let its boundary, *viz.* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, be maintained at temperature zero. Then the required temperature $v(x, y, z, t)$ is such that

(1) $\frac{\delta v}{\delta t}=\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}+\frac{\delta^2 v}{\delta z^2}$, the units being so chosen that the diffusivity is 1,

(2) $v=0$ on the boundary,

(3) $v=f(x, y, z)$ when $t=0$.

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* *Cours de physique mathematique,* Ch. IX.

† *Phil. Trans.,* Vol. 171 (1880).

Thus v can be expressed as a sum of terms of the form

$$Ae^{-\lambda^2 t}\ W(x, y, z),$$

where the normal function W satisfies the equation

$$\frac{\delta^2 W}{\delta x^2}+\frac{\delta^2 W}{\delta y^2}+\frac{\delta^2 W}{\delta z^2}=-\lambda^2 W, \quad \dots \quad \dots \quad (I)$$

and vanishes on the boundary; and the constants A are so chosen that the initial condition (3) is satisfied, so that

$$f(x, y, z)=\Sigma\, A\, W.$$

When $a=b=c$, so that the ellipsoid becomes a sphere of radius a, an appropriate normal function is

$$S_n(\lambda r)\ P_n^m \cos m\,\phi,$$

where $S_n(x)=(-1)^n \sqrt{\frac{\pi}{2}}\ x^{-\frac{1}{2}} J_{n+\frac{1}{2}}(x)$,

$$P_n^m=\sin^m \theta \frac{d^m P_n}{d\,\mu^m},\ \mu \text{ being } \cos\theta,$$

and λ is a root of the equation $S_n(\lambda a)=0$.

Throughout the present paper, I will represent by W_n^m the normal function corresponding to the function

$$S_n(\lambda r)\ P_n^m \cos m\,\phi,$$

and denote W_n^0 by W_n.

I proceed now to obtain the functions W of various types.

W_0 for ellipsoid of revolution.

3. Let e denote the eccentricity of the ellipsoid; then neglecting e^4 and higher powers, the equation of the ellipsoid can be written as

$$r=a\,\{1-\tfrac{1}{3}e^2+\tfrac{1}{3}e^2 P_2(\cos\theta)\},$$

$$i.e.,\ r=\alpha\left(1+\epsilon\ P_2(\cos\theta)\right),$$

where $\alpha=a\,(1-\frac{1}{3}e^2)$ and $\epsilon=\frac{1}{3}e^2$.

Now assume that

$$W_0=S_0(\lambda r)+\epsilon \sum_{t=1}^{\infty} M_t\ S_t(\lambda r)\ P_t\ (\cos\theta),$$

M_t being an unknown constant to be determined.

Then, evidently, W_0 satisfies the partial differential equation (1); and to satisfy the boundary condition we must have

$$o = S_0\ (\lambda a) + \epsilon a \frac{\delta S_0\ (\lambda a)}{\delta a} P_2 + \epsilon \sum_{t=1}^{\infty} M_t\ S_t\ (\lambda a)\ P_t\ (\cos\theta),$$

since ϵ^2 and higher powers are neglected.

This equation must hold for all values of $\cos\theta$. Therefore, equating to zero the coefficients of the various zonal harmonics, we get

$$S_0\ (\lambda a) = o, \qquad (1)$$

$$M_2\ S_2\ (\lambda a) + a \frac{\delta S_0\ (\lambda a)}{\delta a} = o\ ; \qquad (2)$$

and all the other M's are zero.

Hence the required expression for W_0, in terms of a and e, is

$$W_0 = S_0\ (\lambda r) - \tfrac{1}{3} e^2 \frac{a \dfrac{\delta S_0\ (\lambda a)}{\delta a}}{S_2\ (\lambda a)} S_2\ (\lambda r)\ P_2\ (\cos\theta),$$

where λ is given by the equation (1).

But the general solution of the equation (1) is known to be

$$\lambda a = i\pi,$$

i being any integer.

Hence $$\lambda a = i\pi + \tfrac{1}{3}\, i\pi\, e^2.$$

4. In order to obtain a closer approximation to W_0, we will retain e^4 and *neglect the sixth and higher powers.* Thus the equation of the ellipsoid is

$$r = a \left[(1 - \tfrac{1}{6}e^2 - \tfrac{3}{16}e^4) + (\tfrac{1}{3}e^2 + \tfrac{1}{21}e^4)\ P_2\ (\cos\theta) + \tfrac{3}{35}e^4\, P_4\ (\cos\theta) \right],$$

i.e., $$r = \beta \left[1 + \sigma\ P_2\ (\cos\theta) + \tau\ P_4\ (\cos\theta) \right],$$

where $$\beta = a\ (1 - \tfrac{1}{6}e^2 - \tfrac{3}{15}e^4),$$

$$\sigma = \tfrac{1}{3}e^2 + \tfrac{10}{63}e^4,$$

and $$\tau = \tfrac{3}{35}e^4.$$

Let us assume that

$$W_0 = S_0(\lambda r) - \tfrac{1}{3}e^2 \frac{a\dfrac{\delta S_0(\lambda a)}{\delta a}}{S_2(\lambda a)} S_2(\lambda r).P_2(\cos\theta)$$

$$+ \tau \sum_{t=1}^{\infty} N_t\, S_t(\lambda r)\, P_t(\cos\theta),$$

N_t being an unknown constant to be determined.

Now, expanding by Taylor's Theorem, we have

$$S_0(\lambda r) = S_0(\lambda\beta) + \beta \frac{\delta S_0(\lambda\beta)}{\delta\beta}(\sigma P_2 + \tau P_4)$$

$$+ \frac{1}{2!}\beta^2 \frac{\delta^2 S_0(\lambda\beta)}{\delta\beta^2}(\sigma P_2 + \tau P_4)^2 + \ldots,$$

and

$$S_2(\lambda r) = S_2(\lambda\beta) + \beta \frac{\delta S_2(\lambda\beta)}{\delta\beta}(\sigma P_2 + \tau P_4) + \ldots,$$

when $r = \beta(1 + \sigma P_2 + \tau P_4)$.

Again $(P_2)^2 = \frac{18}{35}P_4 + \frac{2}{7}P_2 + \frac{1}{5}$.

Hence we must have

$$0 = S_0(\lambda\beta) + \beta\frac{\delta S_0(\lambda\beta)}{\delta\beta}\left(\sigma P_2 + \tau P_4\right) + \frac{\sigma^2}{2}\beta^2\frac{\delta^2 S_0(\lambda\beta)}{\delta\beta^2}\left[\tfrac{18}{35}P_4 + \tfrac{2}{7}P_2 + \tfrac{1}{5}\right]$$

$$-\tfrac{1}{3}e^2\frac{a\dfrac{\delta S_0(\lambda a)}{\delta a}}{S_2(\lambda a)} S_2(\lambda\beta)P_2 - \tfrac{1}{3}e^2\sigma\frac{a\dfrac{\delta S_0(\lambda a)}{\delta a}}{S_2(\lambda a)}\,\beta\frac{\delta S_2(\lambda\beta)}{\delta\beta}\left[\tfrac{18}{35}P_4 + \tfrac{2}{7}P_2 + \tfrac{1}{5}\right],$$

$$+\tau\sum_{t=1}^{\infty} N_t S_t(\lambda\beta)P_t.$$

This equation must hold for all values of $\cos\theta$. Therefore, equating to zero the coefficients of the various zonal harmonics, we get,

$$S_0(\lambda\beta) + \tfrac{1}{10}\sigma^2\beta^2\frac{\delta^2 S_0(\lambda\beta)}{\delta\beta^2} - \tfrac{1}{15}e^2\sigma\frac{a\dfrac{\delta S_0(\lambda a)}{\delta a}}{S_2(\lambda a)}\,\beta\frac{\delta S_2(\lambda\beta)}{\delta\beta} = 0, \quad \ldots \quad (1)$$

$$\tau N_2 S_2(\lambda\beta) + \sigma\beta\frac{\delta S_0(\lambda\beta)}{\delta\beta} + \tfrac{1}{7}\sigma^2\beta^2\frac{\delta^2 S_0(\lambda\beta)}{\delta\beta^2} - \tfrac{1}{3}e^2\frac{a\dfrac{\delta S_0(\lambda a)}{\delta a}}{S_2(\lambda a)}\cdot S_2(\lambda\beta)$$

$$- \tfrac{2}{21}e^2\sigma\frac{a\dfrac{\delta S_0(\lambda a)}{\delta a}}{S_2(\lambda a)}\cdot\beta\frac{\delta S_2(\lambda\beta)}{\delta\beta} = 0, \qquad \ldots \quad (2)$$

$$\tau N_4 S_4(\lambda\beta) + \tau\beta\frac{\delta S_0(\lambda\beta)}{\delta\beta} + \tfrac{9}{35}\sigma^2\beta^2\frac{\delta^2 S_0(\lambda\beta)}{\delta\beta^2}$$

$$- \tfrac{8}{35}e^2\sigma\frac{a\dfrac{\delta S_0(\lambda a)}{\delta a}}{S_2(\lambda a)}\beta\frac{\delta S_2(\lambda\beta)}{\delta\beta} = 0; \qquad \ldots \quad (3)$$

and all the other N's are zero.

Now, on substituting the values of β, σ and τ, in terms of a and e, the above equations take the forms

$$S_0(\lambda a) - (\tfrac{1}{3}e^2 + \tfrac{2}{15}e^4)a\frac{\delta S_0(\lambda a)}{\delta a} + \tfrac{1}{15}e^4 a^2\frac{\delta^2 S_0(\lambda a)}{\delta a^2}$$

$$- \tfrac{1}{45}e^4\frac{a\dfrac{\delta S_0(\lambda a)}{\delta a}}{S_2(\lambda a)}\, a\frac{\delta S_2(\lambda a)}{\delta a} = 0, \qquad \ldots \quad (4)$$

$$\tfrac{3}{35}N_2 S_2(\lambda a) + \tfrac{1}{21}a\frac{\delta S_0(\lambda a)}{\delta a} - \tfrac{2}{21}a^2\frac{\delta^2 S_0(\lambda a)}{\delta a^2}$$

$$+ \tfrac{8}{63}\cdot\frac{a\dfrac{\delta S_0(\lambda a)}{\delta a}}{S_2(\lambda a)}\cdot a\frac{\delta S_2(\lambda a)}{\delta a} = 0, \qquad \ldots \quad (5)$$

$$\tfrac{8}{35}N_4 S_4(\lambda a) + \tfrac{3}{35}a\frac{\delta S_0(\lambda a)}{\delta a} + \tfrac{1}{35}a^2\frac{\delta^2 S_0(\lambda a)}{\delta a^2}$$

$$- \tfrac{3}{35}\frac{a\dfrac{\delta S_0(\lambda a)}{\delta a}}{S_2(\lambda a)}\cdot a\frac{\delta S_2(\lambda a)}{\delta a} = 0, \qquad \ldots \quad (6)$$

Hence the required expression for W_0 is,

$$W_0 = S_0(\lambda r) - \tfrac{1}{3}e^2\frac{a\dfrac{\delta S_0(\lambda a)}{\delta a}}{S_2(\lambda a)}S_2(\lambda r)P_2(\cos\theta) + \tfrac{3}{35}e^4 N_2 S_2(\lambda r)P_2(\cos\theta)$$

$$+ \tfrac{8}{35}e^4 N_4 S_4(\lambda r)P_4(\cos\theta),$$

the values of N_2 and N_4 being given by the equations (5) and (6) and the value of λ being determined from the equation (4).

5. I will now proceed with the solution of the equation (4) of the preceding article. If we neglect e altogether, the equation reduces to $S_o(\lambda a)=0$, whose roots are given by $\lambda a=i\pi$. Therefore let the full value of λa be

$$i\pi+l_1e^2+l_2e^4+\ \ldots,$$

l_1, l_2 being unknown quantities which are to be determined.

Now S_n can be expanded in a series containing a finite number of terms so that

$$S_n(x)=\left[\frac{1}{x}-\frac{n'(n'-1')}{1\cdot 2}\cdot\frac{1}{2^2x^3}+\frac{n'(n'-1')(n'-2')(n'-3')}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{2^4x^5}-\ \ldots\right]$$

$$\times\sin\left(x+\frac{n\pi}{2}\right)$$

$$+\left[\frac{n'}{1}\cdot\frac{1}{2x^2}-\frac{n'(n'-1')(n'-2')}{1\cdot 2\cdot 3}\cdot\frac{1}{2^3x^4}+\ldots\right]\cos\left(x+\frac{n\pi}{2}\right),$$

where n' stands for $n(n+1)$ and $(n'-t')$ for $(n-t)\ (n+t+1)$.

Hence we have the following:—

$$S_o\ (\lambda a)=\frac{l_1e^2}{i\pi}\ \cos i\pi\ +\left(\frac{l_2}{i\pi}\ -\frac{l_1{}^2}{i^2\pi^2}\right)\cos i\pi\ e^4\ +\ \ldots$$

$$S_2\ (\lambda a)=-\ \frac{3}{i^2\pi^2}\ \cos i\pi\ +\ \ldots$$

$$a\ \frac{\delta S_o(\lambda a)}{\delta a}\ =\cos i\pi-\frac{l_1}{i\pi}\cos i\pi\ e^2\ +\ \ldots$$

$$a^2\ \frac{\delta^2 S_o(\lambda a)}{\delta a^2}\ =-\ 2\cos i\pi\ +\ \ldots$$

$$a\ \frac{\delta S_2(\lambda a)}{\delta a}\ =-\left(1-\frac{9}{i^2\pi^2}\right)\cos i\pi\ +\ \ldots$$

$$a\ \frac{\delta S_o(\lambda a)}{\delta a}\cdot a\ \frac{\delta S_2(\lambda a)}{\delta a}\ \Big/\ S_2(\lambda a)=\tfrac{1}{3}\ (i^2\pi^2-9)\cos i\pi\ +\ \ldots$$

Therefore, substituting the above expressions for $S_o\cdot(\lambda a)$, etc. in the equation (4) of the preceding article and equating to zero the coefficients of e^2 and e^4, we obtain finally

$$\lambda a=i\pi+\tfrac{1}{3}\ i\pi\ e^2+\ \frac{i\pi}{135}\ (i^2\pi^2+27)\ e^4\ +\ \ldots$$

Comparison with the results of Niven.

6. By neglecting e^6 and higher powers, Professor C. Niven has obtained an expression* for the parameter λ, *viz.*,

$$\lambda a = i\pi + \frac{i\pi}{3} e^2 + \frac{i\pi}{405} (i^2\pi^2 + 27) e^4 + \ldots,$$

which differs from that obtained by me in the preceding article only in so far as the coefficient of e^4 is concerned. After carefully going through Professor Niven's calculations, I find that the mistake of Professor Niven must be attributed to some inadvertence on his part, as, by repeating his process, I get the correct result.

From the identity of the values of λ, it follows at once that my expression for W_0 and Professor Niven's must be identical. For, as is well-known, for the same value of λ there cannot be two different solutions, of the same type, of the equation

$$\frac{\delta^2 W_0}{\delta x^2} + \frac{\delta^2 W_0}{\delta y^2} + \frac{\delta^2 W_0}{\delta z^2} + \lambda^2 W_0 = o$$

both of which vanish on the surface of the ellipsoid.

Closer approximation to W_0 for ellipsoid of revolution.

7. For a still closer approximation, we will now retain e^6 also and neglect the eighth and higher powers. Then the equation of the ellipsoid is

$$r = c_0 + c_2 P_2 + c_4 P_4 + c_6 P_6,$$

where $$c_0/a = 1 - \tfrac{1}{3}e^2 - \tfrac{2}{15}e^4 - \tfrac{8}{105}e^6,$$

$$c_2/a = \tfrac{1}{3}e^2 + \tfrac{1}{21}e^4,$$

$$c_4/a = \tfrac{3}{35}e^4 - \tfrac{2}{35}e^6,$$

and $$c_6/a = \tfrac{8}{231}e^6.$$

Let us now assume that

$$W_0 = S_0(\lambda r) + A_2 S_2(\lambda r) P_2 + A_4 S_4(\lambda r) P_4 + e^6 \sum_{t=1}^{\infty} M_t S_t(\lambda r) P_t(\mu),$$

* *Loc. cit.*, p. 145.

where M_t is an unknown constant to be determined and A_2, A_4 are given by

$$A_2 = -(\tfrac{1}{3}e^2 + \tfrac{1}{21}e^4)\,\frac{a\dfrac{\delta S_0}{\delta a}}{S_2} + \tfrac{2}{21}e^4\,\frac{a^2\dfrac{\delta^2 S_0}{\delta a^2}}{S_2} - \tfrac{5}{63}e^4\,\frac{a\dfrac{\delta S_0}{\delta a}\cdot a\dfrac{\delta S_2}{\delta a}}{(S_2)^2},$$

$$\text{and } A_4 = -e^4\left\{\tfrac{3}{35}\,a\frac{\delta S_0}{\delta a} + \tfrac{1}{35}\,a^2\,\frac{\delta^2 S_0}{\delta a^2} - \tfrac{3}{35}\,\frac{a\dfrac{\delta S_0}{\delta a}\cdot a\dfrac{\delta S_2}{\delta a}}{S_2}\right\}\Big/ S_4,$$

the equations determined in Art 4. This expression for W_0 evidently satisfies the differential equation (I). To satisfy the boundary condition we must have

$$0 = S_0\,(\lambda c_0) + \frac{\delta S_0(\lambda c_0)}{\delta c_0}\,(c_2P_2 + c_4P_4 + c_6P_6)$$

$$+ \frac{1}{2!}\,\frac{\delta^2 S_0(\lambda c_0)}{\delta c_0^2}(c_2P_2 + c_4P_4 + c_6P_6)^2$$

$$+ \frac{1}{3!}\,\frac{\delta^3 S_0(\lambda c_0)}{\delta c_0^3}\,(c_2P_2 + c_4P_4 + c_6P_6)^3 + \quad \cdots$$

$$+ A_2P_2\left[S_2\,(\lambda c_0) + \frac{\delta S_2(\lambda c_0)}{\delta c_0}\,(c_2P_2 + c_4P_4 + c_6P_6)\right.$$

$$\left. + \frac{1}{2!}\,\frac{\delta^2 S_2(\lambda c_0)}{\delta c_0^2}\,(c_2P_2 + c_4P_4 + c_6P_6)^2 + \quad \cdots\right]$$

$$+ A_4P_4\left[S_4\,(\lambda c_0) + \frac{\delta S_4\,(\lambda c_0)}{\delta c_0}\,(c_2P_2 + c_4P_4 + c_6P_6) + \quad \cdots\right]$$

$$+ e^6 \sum_{t=1}^{\infty} M_t\,S_t\,(\lambda c_0)\,P_t.$$

Again

$$P_2P_4 = \tfrac{5}{11}\,P_6 + \tfrac{20}{77}\,P_4 + \tfrac{2}{7}\,P_2,$$

$$P_2 \times P_2 \times P_2 = \tfrac{18}{77}\,P_6 + \tfrac{108}{385}\,P_4 + \tfrac{3}{7}\,P_2 + \tfrac{2}{35}.$$

Hence, to our degree of approximation, we must have

$$0 = S_0\,\;(\lambda c_0) + \frac{\delta S_0(\lambda c_0)}{\delta c_0}\,\;(c_2P_2 + c_4P_4 + c_6P_6)$$

$$+ \frac{1}{2!}(c_2)^2 \frac{\delta^2 S_0(\lambda c_0)}{\delta c_0^{\,2}} \left[\tfrac{18}{35} P_4 + \tfrac{2}{7} P_2 + \tfrac{1}{5} \right]$$

$$+ \frac{1}{2!} c_2 c_4 \frac{\delta^2 S_0(\lambda c_0)}{\delta c_0^{\,2}} \, 2 \left[\tfrac{5}{11} P_6 + \tfrac{20}{77} P_4 + \tfrac{2}{7} P_2 \right]$$

$$+ \frac{(c_2)^3}{3!} \frac{\delta^3 S_0(\lambda c_0)}{\delta c_0^{\,3}} \left[\tfrac{18}{77} P_6 + \tfrac{108}{385} P_4 + \tfrac{3}{7} P_2 + \tfrac{2}{35} \right]$$

$$+ A_2 S_2(\lambda c_0) P_2 + A_2 c_2 \frac{\delta S_2(\lambda c_0)}{\delta c_0} \left[\tfrac{18}{35} P_4 + \tfrac{2}{7} P_2 + \tfrac{1}{5} \right]$$

$$+ A_2 c_4 \frac{\delta S_2(\lambda c_0)}{\delta c_0} \left[\tfrac{5}{11} P_6 + \tfrac{20}{77} P_4 + \tfrac{2}{7} P_2 \right]$$

$$+ A_2 \frac{(c_2)^2}{2!} \frac{\delta^2 S_2(\lambda c_0)}{\delta c_0^{\,2}} \left[\tfrac{18}{77} P_6 + \tfrac{108}{385} P_4 + \tfrac{3}{7} P_2 + \tfrac{2}{35} \right]$$

$$+ A_4 S_4(\lambda c_0) P_4 + A_4 c_2 \frac{\delta S_4(\lambda c_0)}{\delta c_0} \left[\tfrac{5}{11} P_6 + \tfrac{20}{77} P_4 + \tfrac{2}{7} P_2 \right]$$

$$+ e^6 \sum_{t=1}^{\infty} M_t S_t(\lambda c_0) P_t.$$

This equation must hold for all values of $\cos\theta$. Hence equating to zero the coefficients of the various zonal harmonics, we get

$$S_0(\lambda c_0) + \frac{(c_2)^2}{10} \frac{\delta^2 S_0(\lambda c_0)}{\delta c_0^{\,2}} + \frac{(c_2)^3}{105} \frac{\delta^3 S_0(\lambda c_0)}{\delta c_0^{\,3}}$$

$$+ A_2 \left[\frac{c_2}{5} \frac{\delta S_2(\lambda c_0)}{\delta c_0} + \frac{(c_2)^2}{35} \frac{\delta^2 S_2(\lambda c_0)}{\delta c_0^{\,2}} \right] = 0, \quad \ldots \quad (1)$$

$$c_2 \frac{\delta S_0(\lambda c_0)}{\delta c_0} + \frac{(c_2)^2}{7} \frac{\delta^2 S_0(\lambda c_0)}{\delta c_0^{\,2}} + \frac{2 c_2 c_4}{7} \frac{\delta^2 S_0(\lambda c_0)}{\delta c_0^{\,2}}$$

$$+ \frac{(c_2)^3}{14} \frac{\delta^3 S_0(\lambda c_0)}{\delta c_0^{\,3}}$$

$$+ A_2 \left[S_2(\lambda c_0) + \tfrac{2}{7} c_2 \frac{\delta S_2(\lambda c_0)}{\delta c_0} + \tfrac{2}{7} c_4 \frac{\delta S_2(\lambda c_0)}{\delta c_0} \right.$$

$$\left. + \tfrac{3}{14} (c_2)^2 \frac{\delta^2 S_2(\lambda c_0)}{\delta c_0^{\,2}} \right]$$

$$+ \tfrac{2}{7} A_4 c_2 \frac{\delta S_4(\lambda c_0)}{\delta c_0} + e^6 M_2 S_2(\lambda c_0) = 0, \quad \ldots \quad (2)$$

$$c_4 \frac{\delta S_0(\lambda c_0)}{\delta c_0} + \tfrac{9}{35}(c_2)^2 \frac{\delta^2 S_0(\lambda c_0)}{\delta c_0^2} + \tfrac{20}{77} c_2 c_4 \frac{\delta^2 S_0(\lambda c_0)}{\delta c_0^2}$$

$$+ \tfrac{18}{385}(c_2)^3 \frac{\delta^3 S_0(\lambda c_0)}{\delta c_0^3}$$

$$+ A_2 \left[\tfrac{18}{35} c_2 \frac{\delta S_2(\lambda c_0)}{\delta c_0} + \tfrac{20}{77} c_4 \frac{\delta^2 S_0(\lambda c_0)}{\delta c_0^2} \right.$$

$$\left. + \tfrac{64}{385}(c_2)^2 \frac{\delta^2 S_2(\lambda c_0)}{\delta c_0^2} \right]$$

$$+ A_4 \left[S_4(\lambda c_0) + \tfrac{20}{77} c_2 \frac{\delta S_4(\lambda c_0)}{\delta c_0} \right] + e^6 M_4 S_4(\lambda c_0) = 0, \quad \ldots \quad (3)$$

$$c_6 \frac{\delta S_0(\lambda c_0)}{\delta c_0} + \tfrac{5}{11} c_2 c_4 \frac{\delta^2 S_0(\lambda c_0)}{\delta c_0^2} + \tfrac{3}{77}(c_2)^3 \frac{\delta^3 S_0(\lambda c_0)}{\delta c_0^3}$$

$$+ A_2 \left[\tfrac{5}{11} c_4 \frac{\delta S_2(\lambda c_0)}{\delta c_0^2} + \tfrac{2}{77}(c_2)^2 \frac{\delta^2 S_2(\lambda c_0)}{\delta c_0^2} \right]$$

$$+ A_4 \times \tfrac{5}{11} c_2 \frac{\delta S_4(\lambda c_0)}{\delta c_0} + e^6 M_6 S_6(\lambda c_0) = 0, \quad \ldots \quad (4)$$

and all the other M's are zero.

Hence the required expression for W_0 is

$$W_0 = S_0(\lambda r) + (A_2 + e^6 M_2) S_2(\lambda r) P_2 + (A_4 + e^6 M_4) S_4(\lambda r) P_4$$
$$+ e^6 M_6 S_6(\lambda r) P_6,$$

the values of M_2, M_4 and M_6 being given by the equations (2), (3), (4) and the value of λ being determined from the equation (1).

8. I will now proceed to the solution of the equation (1) of the preceding article. On substituting the values of c_0, c_2 and A_2 in terms of a and e and writing D_a for $a\dfrac{\delta}{\delta a}$, the equation (1) takes the form

$$S_0(\lambda a) - \left(\tfrac{1}{3}e^2 + \tfrac{2}{15}e^4 + \tfrac{8}{105}e^6\right) D_a S_0 + \left(\tfrac{1}{15}e^4 + \tfrac{1}{21}e^6\right) D_a^2 S_0$$

$$- \tfrac{1}{105}e^6 D_a^3 S_0 - \left(\tfrac{1}{45}e^4 + \tfrac{2}{315}e^6\right) D_a S_0 \cdot D_a S_2 / S_2$$

$$+ \tfrac{2}{315}e^6 D_a^2 S_0 \cdot D_a S_2 / S_2 - \tfrac{1}{189}e^6 D_a S_0 \cdot D_a S_2 \cdot D_a S_2 / S_2 \cdot S_2$$

$$+ \tfrac{2}{315} D_a S_0 \cdot D_a^2 S_2 / S_2 = 0. \quad \ldots \quad \ldots \quad (5)$$

If we neglect e altogether, the equation reduces to $S_0(\lambda a)=0$, whose roots are given by $\lambda a=i\pi$. Therefore let the full value of λa be

$$i\pi+l_1e^2+l_2e^4+l_3e^6+\quad\ldots\quad,$$

l_1, l_2, l_3, being unknown quantities which are to be determined. Then we have the following:—

$$S_0(\lambda a)=\frac{l_1}{i\pi}\cos i\pi\, e^2+\left(\frac{l_2}{i\pi}-\frac{l_1^2}{i^2\pi^2}\right)e^4\cos i\pi$$

$$+\left(\frac{l_3}{i\pi}-\frac{2l_1l_2}{i^2\pi^2}+\frac{l_1^3}{i^3\pi^3}-\frac{l_1^3}{6i\pi}\right)e^6\cos i\pi+\quad\ldots\quad,$$

$$D_aS_0=\cos i\pi-\frac{l_1}{i\pi}e^2\cos i\pi-\left(\frac{l_2}{i\pi}-\frac{l_1^2}{i^2\pi^2}+\frac{l_1^2}{2}\right)e^4\cos i\pi+\quad\ldots,$$

$$D_a^2S_0=-2\cos i\pi-\left(i\pi-\frac{2}{i\pi}\right)l_1e^2\cos i\pi+\quad\ldots\quad,$$

$$D_a^3S_2=(6-i^2\pi^2)\cos i\pi\quad+\quad\ldots$$

$$D_aS_2=-\left(1-\frac{9}{i^2\pi^2}\right)\cos i\pi+\left(4-\frac{27}{i^2\pi^2}\right)\frac{l_1}{i\pi}\cos i\pi\, e^2+\quad\ldots\quad,$$

$$D_a^2S_2=\left(5-\frac{36}{i^2\pi^2}\right)\cos i\pi+\quad\ldots\quad,$$

$$D_aS_0.\,D_aS_2\big/S_2=\tfrac{1}{3}(i^2\pi^2-9)\cos i\pi$$

$$-\tfrac{1}{9}(i^4\pi^4-5i^2\pi^2-27)\frac{l_1}{i\pi}\cos i\pi\, e^2+\quad\ldots\quad,$$

$$D_a^2S_0.\,D_aS_2\big/S_2=-\tfrac{2}{3}(i^2\pi^2-9)\cos i\pi+\quad\ldots\quad,$$

$$D_aS_0.\,D_aS_2.\,D_aS_2\big/S_2S_2=\tfrac{1}{9}(i^2\pi^2-9)^2\cos i\pi+\quad\ldots\quad,$$

$$D_aS_0.D_a^2S_2\big/S_2=-\tfrac{1}{3}(5\,i^2\pi^2-36)\cos i\pi+\quad\ldots\quad,$$

Therefore substituting the above expressions for S_0 (λa) etc., in the equation (5) and equating to zero the coefficients of e^2, e^4, e^6, we obtain finally,

$$\lambda a = i\pi + \frac{i\pi}{8} e^2 + \frac{i\pi}{135}(i^2\pi^2+27)\; e^4$$

$$+ \frac{i\pi}{9\times 945}(1215 + 99\, i^2\pi^2 - 2i^4\pi^4)\, e^6 + \ldots$$

W_n for ellipsoid of revolution, $n>0$.

9. Let e^4 and higher powers be neglected, so that the equation to the ellipsoid is the same as in Art. 3, *viz.*

$$r = a\{1+\epsilon\, P_2\,(\cos\theta)\}.$$

Assume that

$$W_n = S_n\,(\lambda r)\, P_n\,(\cos\;\theta) + \epsilon \sum_{t=0}^{\infty}{}' \; H_t\; S_t\,(\lambda r)\, P_t\,(\cos\theta),$$

where the H's are unknown constants to be determined and $\sum\limits_{t=0}^{\infty}{}'$

refers to all the values of t except $t=n$. Thus it is evident that W_n satisfies the partial differential equation (I), and it remains to find the values of the constants H's so as to satisfy the boundary condition.

Now, since ϵ^2 and higher powers of ϵ are to be neglected, we have, on putting $r = a\,(1+\epsilon\;P_2)$,

$$S_n\,(\lambda r) = S_n\,(\lambda a) + \epsilon a\; \frac{\delta S_n(\lambda a)}{\delta a}\, P_2\;;$$

also, we have

$$P_2\; P_n = B_{n+2}\; P_{n+2} + C_n\; P_n + D_{n-2}\; P_{n-2},$$

where $$B_{n+2} = \tfrac{3}{2}\,\frac{(n+1)\,(n+2)}{(2n+3)\,(2n+1)},$$

$$C_n = \frac{n(n+1)}{(2n+3)\,(2n-1)},$$

$$D_{n-2} = \tfrac{3}{2}\,\frac{n(n-1)}{(2n+1)\,(2n-1)}.$$

Hence we must have

$$o = S_n(\lambda a)\, P_n + \epsilon a \frac{\delta S_n(\lambda a)}{\delta a}\left[B_{n+2}\ P_{n+2} + C_n\, P_n + D_{n-2}\, P_{n-2}\right]$$

$$+ \epsilon \sum_{t=o}^{\infty}{}' \ H_t\ S_t\ (\lambda a)\ P_t.$$

This must be true for all values of $\cos\theta$. Therefore, equating to zero the co-efficients of the various zonal harmonics, we get

$$S_n\ (\lambda a) + \epsilon a \frac{\delta\, S_n\ (\lambda a)}{\delta a}\ C_n = o, \tag{1}$$

$$H_{n+2}\ S_{n+2}\ (\lambda a) + a\ \frac{\delta\, S_n\ (\lambda a)}{\delta a} B_{n+2} = o, \tag{2}$$

$$H_{n-2}\ S_{n-2}\ (\lambda a) + a\ \frac{\delta\, S_n\ (\lambda a)}{\delta a} D_{n-2} = o, \tag{3}$$

and all the other H's are zero.

Thus the unknown constants are determined and the required expression for W_n, in terms of a and e, is

$$W_n = S_n\ (\lambda r)\, P_n\ (\cos\theta) - \tfrac{1}{3} e^2\ \frac{a\ \dfrac{\delta S_n\ (\lambda a)}{\delta a}}{S_{n+2}\ (\lambda a)}\ S_{n+2}\ (\lambda r)\ P_{n+2}\ (\cos\theta)\ B_{n+2}$$

$$- \tfrac{1}{3} e^2\ \frac{a \dfrac{\delta\, S_n\ (\lambda a)}{\delta a}}{S_{n-2}\ (\lambda a)}\ S_{n-2}\ (\lambda r)\ P_{n-2}\ (\cos\theta)\ D_{n-2}$$

λ being a root of the equation (1).

But, in terms of a and e, this equation is

$$S_n\ (\lambda a) - \frac{n^2 + n - 1}{(2n+3)\ (2n-1)}\ \ e^2\ a\,\frac{\delta\, S_n\ (\lambda a)}{\delta a} = 0. \tag{4}$$

Therefore, if κ be a root of the equation $S_n\ (\kappa) = 0$, the corresponding solution of (4) is given by

$$\lambda\, a = \kappa\ \left\{ 1 + \frac{n^2 + n - 1}{(2n+3)\ (2n-1)} e^2 \right\}.$$

10. In order to obtain a closer approximation to W_n, we will retain e^4 and *neglect the sixth and higher powers*, so that the equation to the ellipsoid is the same as in Art 4, *viz.*

$$r=\beta\,[1+\sigma P_2+\tau P_4].$$

Assume that

$$W_n=S_n(\lambda r)P_n+\frac{1}{3}e^2H_{n+2}S_{n+2}(\lambda r)P_{n+2}+\frac{1}{3}e^2H_{n-2}S_{n-2}(\lambda r)P_{n-2}$$
$$+\tau\sum_{t=o}^{\infty}{}'\,H_t^{(1)}\,S_t(\lambda r)P_t(\mu),$$

where $H_t^{(1)}$ is an unknown constant to be determined and H_{n+2} and H_{n-2} have the values given by the equations (2), (3) of Art 9; also $\sum\limits_{t=o}^{\infty}{}'$ denotes that the summation is to be taken for all values of t from o to ∞, except the value $t=n$. This expression for W_n evidently satisfies the differential equation (I). To satisfy the boundary condition also, we must have

$$O=S_n(\lambda\beta)\,P_n+\beta\,\frac{\delta S_n(\lambda\beta)}{\delta\beta}\,(\sigma P_2+\tau P_4)P_n+\frac{1}{2!}\beta^2\,\frac{\delta^2 S_n(\lambda\beta)}{\delta\beta^2}(\sigma P_2+\tau P_4)^2 P_n$$
$$+\frac{1}{3}e^2H_{n+2}\left[S_{n+2}(\lambda\beta)+\beta\,\frac{\delta S_{n+2}(\lambda\beta)}{\delta\beta}\,\sigma P_2\right]P_{n+2}.$$
$$+\frac{1}{3}e^2H_{n-2}\left[S_{n-2}(\lambda\beta)+\sigma\beta\,\frac{\delta S_{n-2}(\lambda\beta)}{\delta\beta}\,P_2\right]P_{n-2}+\tau\sum_{t=o}^{\infty}{}'\,H_t^{(1)}\,S_t(\lambda\beta)P_t,$$

since e^6 and higher powers are neglected.

Now

$$P_2\times P_2\times P_n=M_{n+4}P_{n+4}+M_{n+2}P_{n+2}+M_nP_n+M_{n-2}P_{n-2}+M_{n-4}P_{n-4},$$

and $P_4P_n=N_{n+4}P_{n+4}+N_{n+2}P_{n+2}+N_nP_n+N_{n-2}P_{n-2}+N_{n-4}P_{n-4}$,

where

$$M_{n+4}=\frac{9}{4}\,\frac{(n+1)(n+2)(n+3)(n+4)}{(2n+1)(2n+3)(2n+5)(2n+7)},$$

$$M_{n+2}=3\,\frac{(n+1)(n+2)(n^2+3n-1)}{(2n-1)(2n+1)(2n+3)(2n+7)},$$

$$M_n=\frac{9}{4}\,\frac{(n+1)^2(n+2)^2}{(2n+1)(2n+3)^2(2n+5)}+\frac{n^2(n+1)^2}{(2n+3)^2(2n-1)^2}$$
$$+\frac{9}{4}\,\frac{n^2(n-1)^2}{(2n-3)(2n-1)^2(2n+1)},$$

$$M_{n-2} = 3\,\frac{n(n-1)(n^2-n-3)}{(2n-5)(2n-1)(2n+1)(2n+3)},$$

$$M_{n-4} = \frac{9}{4}\,\frac{n(n-1)(n-2)(n-3)}{(2n+1)(2n-1)(2n-3)(2n-5)},$$

$$N_{n+4} = \frac{35}{8}\cdot\frac{(n+1)(n+2)(n+3)(n+4)}{(2n+1)(2n+3)(2n+5)(2n+7)},$$

$$N_{n+2} = \frac{5}{2}\cdot\frac{n(n+1)(n+2)(n+3)}{(2n-1)(2n+1)(2n+3)(2n+7)},$$

$$N_n = \frac{9}{4}\,\frac{(n-1)n(n+1)(n+2)}{(2n-3)(2n-1)(2n+3)(2n+5)},$$

$$N_{n-2} = \frac{5}{2}\,\frac{(n-2)(n-1)n(n+1)}{(2n-5)(2n-1)(2n+1)(2n+3)},$$

and $$N_{n-4} = \frac{35}{8}\cdot\frac{(n-3)(n-2)(n-1)n}{(2n-5)(2n-3)(2n-1)(2n+1)},$$

Hence we must have

$$\begin{aligned}
O = {} & S_n(\lambda\beta)P_n + \sigma\beta\,\frac{\delta S_n(\lambda\beta)}{\delta\beta}\,[B_{n+2}P_{n+2} + C_nP_n + D_{n-2}P_{n-2}] \\
& + \tau\beta\,\frac{\delta S_n(\lambda\beta)}{\delta\beta}[N_{n+4}P_{n+4} + N_{n+2}P_{n+2} + N_nP_n + N_{n-2}P_{n-2} + N_{n-4}P_{n-4}] \\
& + \frac{1}{2}\sigma^2\beta^2\,\frac{\delta^2 S_n(\lambda\beta)}{\delta\beta^2}\,[M_{n+4}P_{n+4} + M_{n+2}P_{n+2} + M_nP_n + M_{n-2}P_{n-2} \\
& \qquad + M_{n-4}P_{n-4}] \\
& + \frac{1}{3}\,e^2 H_{n+2}S_{n+2}(\lambda\beta)P_{n+2} \\
& + \frac{1}{3}\,e^2\sigma H_{n+2}\beta\,\frac{\delta S_{n+2}(\lambda\beta)}{\delta\beta}\,[B_{n+4}P_{n+4} + C_{n+2}P_{n+2} + D_nP_n] \\
& + \frac{1}{3}\,e^2 H_{n-2}S_{n-2}(\lambda\beta)P_{n-2} \\
& + \frac{1}{3}\,e^2\sigma H_{n-2}\beta\,\frac{\delta S_{n-2}(\lambda\beta)}{\delta\beta}\,[B_nP_n + C_{n-2}P_{n-2} + D_{n-4}P_{n-4}] \\
& + \tau\sum_{t=o}^{\infty}{}'\ \overset{(1)}{H_t}\,S_t\,(\lambda\beta)P_t.
\end{aligned}$$

This equation must hold for all values of cos θ. Hence equating to zero the co-efficients of the various zonal harmonics, we get

$$\tau \, H^{(1)}_{n+4} S_{n+4} (\lambda\beta) + \tfrac{1}{3}e^2\sigma \, H_{n+2} \, B_{n+4} \, \beta \frac{\delta S_{n+2} (\lambda\beta)}{\delta\beta}$$

$$+ \tfrac{1}{2}\sigma^2\beta^2 \, M_{n+4} \frac{\delta^2 S_n(\lambda\beta)}{\delta\beta^2} + \tau\beta \frac{\delta S_n(\lambda\beta)}{\delta\beta} N_{n+4} = 0, \quad \ldots (1)$$

$$\tau \, H^{(1)}_{n+2} S_{n+2} (\lambda\beta) + \tfrac{1}{3}e^2\sigma\beta \frac{\delta S_{n+2} (\lambda\beta)}{\delta\beta} H_{n+2} \, C_{n+2}$$

$$+ \tfrac{1}{3}e^2 H_{n+2} \, S_{n+2} (\lambda\beta) + \tfrac{1}{2}\sigma^2\beta^2 \frac{\delta^2 S_n(\lambda\beta)}{\delta\beta^2} M_{n+2}$$

$$+ \tau\beta \frac{\delta S_n(\lambda\beta)}{\delta\beta} N_{n+2} + \sigma\beta \frac{\delta S_n(\lambda\beta)}{\delta\beta} B_{n+2} = 0, \quad \ldots (2)$$

$$S_n (\lambda\beta) + \sigma\beta \frac{\delta S_n (\lambda\beta)}{\delta\beta} C_n + \tau\beta \frac{\delta S_n(\lambda\beta)}{\delta\beta} N_n$$

$$+ \tfrac{1}{2}\sigma^2\beta^2 \frac{\delta^2 S_n(\lambda\beta)}{\delta\beta^2} M_n + \tfrac{1}{3}e^2\sigma\beta \frac{\delta S_{n+2}(\lambda\beta)}{\delta\beta} H_{n+2} \, D_n$$

$$+ \tfrac{1}{3}e^2\sigma\beta \frac{\delta S_{n-2} (\lambda\beta)}{\delta\beta} H_{n-2} \, B_n = 0, \quad \ldots \quad \ldots (3)$$

$$\tau H^{(1)}_{n-2} \, S_{n-2} (\lambda\beta) + \tfrac{1}{3}e^2\sigma\beta \frac{\delta S_{n-2}(\lambda\beta)}{\delta\beta} H_{n-2} \, C_{n-2}$$

$$+ \tfrac{1}{3}e^2 S_{n-2} (\lambda\beta) \, H_{n-2} + \tfrac{1}{2}\sigma^2\beta^2 \frac{\delta^2 S_n(\lambda\beta)}{\delta\beta^2} M_{n-2}$$

$$+ [\tau \, N_{n-2} + \sigma D_{n-2}] \, \beta \frac{\delta S_n (\lambda\beta)}{\delta\beta} = 0, \quad \ldots \quad (4)$$

$$\tau H^{(1)}_{n-4} \, S_{n-4} (\lambda\beta) + \frac{\sigma}{3} e^2 \beta \frac{\delta S_{n-2} (\lambda\beta)}{\delta\beta} H_{n-2} \, D_{n-4}$$

$$+ \tfrac{1}{2}\sigma^2\beta^2 \frac{\delta^2 S^n(\lambda\beta)}{\delta\beta^2} M_{n-4} + \tau\beta \frac{\delta S_n(\lambda\beta)}{\delta\beta} N_{n-4} = 0 \, ; \ldots (5)$$

all the other $H^{(1)}$'s are zero.

Thus the unknown constants are determined and the required expression for W_n is given by

$$W_n = S_n(\lambda r) P_n + \left[\frac{1}{3}e^2 H_{n+2} + \tau \overset{(1)}{H}_{n+2}\right] S_{n+2}(\lambda r) P_{n+2}$$

$$+ \tau \overset{(1)}{H}_{n+4} S_{n+4}(\lambda r) P_{n+4} + \tau \overset{(1)}{H}_{n-4} S_{n-4}(\lambda r) P_{n-4}$$

$$+ \left[\frac{1}{3}e^2 H_{n-2} + \tau \overset{(1)}{H}_{n-2}\right] S_{n-2}(\lambda r) P_{n-2}, \quad \ldots \quad (6)$$

where $\overset{(1)}{H}$'s are given by the equations (1), (2), (4) and (5) and λ is a root of the equation (3).

In terms of a and e, this equation becomes

$$S_n(\lambda a) - \frac{1}{3}e^2(1-C_n)\, a\frac{\delta S_n}{\delta a} + e^4\left(\frac{1}{21}C_n + \frac{3}{35}N_n - \frac{2}{15}\right) a\frac{\delta S_n}{\delta a}$$

$$+\frac{1}{18}e^4(M_n - 2C_n + 1)\, a^2 \frac{\delta^2 S_n}{\delta a^2} + \frac{1}{9}e^4 D_n H_{n+2}\, a\frac{\delta S_{n+2}}{\delta a}$$

$$+ \frac{1}{9}e^4 B_n H_{n-2}\, a\frac{\delta S_{n-2}}{\delta a} = 0. \quad \ldots \quad \ldots \quad (7)$$

W_n^m for ellipsoid of revolution.

11. Let e^4 *and higher powers be neglected*, and assume that

$$W_n^m = S_n(\lambda r) P_n^m(\cos\theta) \cos m\phi + \epsilon \sum_{t=m}^{\infty}{}' I_t^m S_t(\lambda r) P_t^m(\cos\theta) \cos m\phi,$$

where $\sum_{t=m}^{\infty}{}'$ refers to all values of t from m up to ∞ except the value $t=n$, and I_t^m is an unknown constant to be determined. Then it is evident that W_n^m satisfies the partial differential equation (I). Now, putting

$$r = a(1+\epsilon P_2),$$

we get

$$S_n(\lambda r) = S_n(\lambda a) + \epsilon a \frac{\delta S_n(\lambda a)}{\delta a} P_2;$$

and also

$$P_2 \, P_n^m = B_{n+2}^m \, P_{n+2}^m + C_n^m \, P_n^m + D_{n-2}^m \, P_{n-2}^m,$$

where

$$B_{n+2}^m = \frac{3}{2} \cdot \frac{(n-m+1)\,(n-m+2)}{(2n+3)\,(2n+1)},$$

$$C_n^m = \frac{n(n+1)-3m^2}{(2n+3)\,(2n-1)},$$

$$D_{n-2}^m = \frac{3}{2} \cdot \frac{(n+m)(n+m-1)}{(2n+1)(2n-1)}.$$

Hence, from the boundary condition, we must have

$$o = S_n\,(\lambda a)\,P_n^m\,(\cos\theta)\cos m\phi + \epsilon a\,\frac{\delta\,S_n\,(\lambda a)}{\delta a}\Big[\,B_{n+2}^m\,P_{n+2}^m + C_n^m\,P_n^m$$

$$+ D_{n-2}^m\,P_{n-2}^m\Big]\cos m\phi + \epsilon\sum_{t=m}^{\infty}{}'\;I_t^m\;S_t(\lambda a)\;P_t^m\,(\cos\theta)\cos m\phi.$$

Therefore, equating to zero the co-efficients of the various surface harmonics, we get

$$S_n\,(\lambda a) + \epsilon a\,\frac{\delta\,S_n\,(\lambda a)}{\delta a}\,C_n^m = o, \qquad (1)$$

$$I_{n+2}^m\;S_{n+2}\,(\lambda a) + a\,\frac{\delta\,S_n\,(\lambda a)}{\delta a}\,B_{n+2}^m = o, \qquad (2)$$

$$I_{n-2}^m\;S_{n-2}\,(\lambda a) + a\,\frac{\delta\,S_n\,(\lambda a)}{\delta a}\,D_{n-2}^m = o; \qquad (3)$$

and all the other I_t^m's are zero.

Thus the required expression for W_n^m, in terms of a and e, is

$$\bar{W}_n^m = \Big[\,S_n\,(\lambda r)\,P_n^m\,(\cos\theta) - \frac{1}{3}e^2\,\frac{a\dfrac{\delta\,S_n(\lambda a)}{\delta a}}{S_{n+2}\,(\lambda a)}\,S_{n+2}\,(\lambda r)\,P_{n+2}^m(\cos\theta)\,B_{n+2}^m$$

$$-\frac{1}{3}e^2\,\frac{a\dfrac{\delta\,S_n\,(\lambda a)}{\delta a}}{S_{n-2}\,(\lambda a)}\,S_{n-2}\,(\lambda r)\,P_{n-2}^m\,(\cos\theta)\,D_{n-2}^m\Big]\cos m\phi$$

where λ is a root of the equation (1).

But, expressed in terms of a and e, this equation becomes

$$S_n(\lambda a) - \frac{(n^2+n-1)+m^2}{(2n+3)(2n-1)} e^2 a \frac{\delta S_n(\lambda a)}{\delta a} = o. \qquad (4)$$

Therefore, we obtain

$$\lambda a = k\left\{1 + \frac{(n^2+n-1)+m^2}{(2n+3)(2n-1)} e^2\right\},$$

corresponding to the root k of $S_n(r) = o$

12. Let us now *retain* e^4 *and neglect* e^6 *and higher powers,* so that the equation of the ellipsoid will be taken to be the same as in Art. 4, *viz.*

$$r = \beta\,[1 + \sigma P_2 + \tau P_4]$$

Then assume that

$$W_n^m = S_n(\lambda r) P_n^m \cos m\phi + \frac{1}{3} e^2 I_{n+2}^m S_{n+2}(\lambda r) P_{n+2}^m \cos m\phi$$

$$+\frac{1}{3} e^2 I_{n-2}^m S_{n-2}(\lambda r) P_{n-}^m \cos m\phi + \tau \sum_{t=m}^{\infty}{}' I'^m_t S_t(\lambda r) P_t^m \cos m\phi$$

where I'^m_t is the unknown constant to be determined and I_{n+2}^m and I_{n-2}^m have the values given by the equations (2), (3) of the preceding article; and $\sum_{t=m}^{\infty}{}'$ refers to all values of t from m to ∞ except the value $t = n$.

This value of W_n^m evidently satisfies the differential equation (I). To satisfy the boundary condition we must have

$$o = S_n(\lambda\beta) P_n^m + \beta \frac{\delta S_n(\lambda\beta)}{\delta\beta}(\sigma P_2 + \tau P_4) P_n^m$$

$$+\frac{1}{2!}\beta^2 \frac{\delta^2 S_n(\lambda\beta)}{\delta\beta^2}(\sigma P_2 + \tau P_4)^2 P_n^m$$

$$+\frac{1}{3}e^2 I_{n+2}^m\left[S_{n+2}(\lambda\beta) + \beta\frac{\delta S_{n+2}(\lambda\beta)}{\delta\beta}\sigma P_2\right] P_{n+2}^m$$

$$+\frac{1}{3}e^2 I^m_{n-2}\left[S_{n-2}(\lambda\beta)+\beta\,\frac{\delta\, S_{n-2}(\lambda\beta)}{\delta\beta}\;\sigma P_2\right]P^m_{n-2}$$

$$+\,\tau\sum_{t=m}^{\infty}{}' \;I'^m_t\; S_t(\lambda\beta)\; P^m_t,$$

since e^6 and higher powers of e are neglected.

Again

$$P_2\times P_2\times P^m_n=M^m_{n+4}\,P^m_{n+4}+M^m_{n+2}\,P^m_{n+2}+M^m_n\,P^m_n+M^m_{n-2}\,P^m_{n-2}$$
$$+\,M^m_{n-4}\,P^m_{n-4},$$

$$P_4\times P^m_n=N^m_{n+4}\,P^m_{n+4}+N^m_{n+2}\,P^m_{n+2}+N^m_n\,P^m_n+N^m_{n-2}\,P^m_{n-2}$$
$$+\,N^m_{n-4}\,P^m_{n-4},$$

where

$$M^m_{n+4}=\frac{9}{4}\cdot\;\frac{(n-m+1)(n-m+2)(n-m+3)(n-m+4)}{(2n+1)(2n+3)(2n+5)(2n+7)},$$

$$M^m_{n+2}=3.\;\;\frac{(n-m+1)(n-m+2)}{(2n-1)(2n+1)(2n+3)(2n+7)}\left[\,(n^2+3n-1)-3m^2\right],$$

$$M^m_n=\frac{9}{4}\cdot\;\frac{(n-m+1)(n-m+2)(n+m+1)(n+m+2)}{(2n+1)(2n+3)^2(2n+5)}$$
$$+\left\{\frac{n(n+1)-3m^2}{(2n-1)(2n+3)}\right\}^2$$
$$+\;\frac{9}{4}\cdot\;\frac{(n+m)(n+m-1)(n-m)(n-m-1)}{(2n+1)(2n-1)^2(2n-3)},$$

$$M^m_{n-2}=3.\;\;\frac{(n+m)(n+m-1)}{(2n-5)(2n-1)(2n+1)(2n+3)}\left[(n^2-n-3)-3m^2\right],$$

$$M^m_{n-4}=\frac{9}{4}\cdot\;\frac{(n+m)(n+m-1)(n+m-2)(n+m-3)}{(2n+1)(2n-1)(2n-3)(2n-5)};$$

and the values of N^m_{n+4} etc., ... can be easily calculated from the following general formula of Adams for the product of any two

Laplace's co-efficients* :—If we have

$$R_n^m = P_n^m \cos m\phi, \quad R_q^p = P_q^p \cos p\phi,$$

then

$$2\, R_n^m\, R_q^p$$

$$= \Sigma\, (-1)^r\, (2n+2q-4r+1)\, R_{n+q-2r}^{m+p}$$

$$\times \frac{(n+q-r)!\ (2q-2r)!\ (n+m)!}{r!\ (q-r)!\ (n-r)!\ (2n+2q-2r+1)!\ (q-p)!}$$

$$\times \left[\Sigma\, (-1)^s \frac{(q-p)!\ (q+p+s)!\ (2n-2r+q-p-s)!}{s!\ (q-p-s)!\ (q+p-2r+s)!\ (n+m-s)!} \right]$$

$$+ \Sigma\, (-1)^r\, (2n+2q-4r+1)\, R_{n+q-2r}^{m-p}$$

$$\times \frac{(n+q-r)!\ (2q-2r)!\ (n+m)!\ (n-m+q+p-2r)!}{r!\ (q-r)!\ (n-r)!\ (2n+2q-2r+1)!\ (n+m+q-p-2r)!\ (q-p)!}$$

$$\times \left[\Sigma\, (-1)^s \frac{(q-p)!\ (2n-2r+q-p-s)!\ (q+p+s)!}{s!\ (q-p-s)!\ (q+p-2r+s)!\ (n-m-s)!} \right],$$

where r has all values from o to $2q$, and s has all values o to $q-p$.

Hence on the surface we must have

$$o = S_n(\lambda\beta) P_n^m + \sigma\beta \frac{\delta S_n(\lambda\beta)}{\delta\beta} \left[B_{n+2}^m\ P_{n+2}^m + C_n^m\ P_n^m + D_{n-2}^m\ P_{n-2}^m \right]$$

$$+ \tau\beta \frac{\delta S_n(\lambda\beta)}{\delta\beta} \left[N_{n+4}^m\, P_{n+4}^m + N_{n+2}^m\, P_{n+2}^m + N_n^m\, P_n^m + N_{n-2}^m\, P_{n-2}^m \right.$$

$$\left. + N_{n-4}^m\ P_{n-4}^m \right]$$

$$+ \frac{1}{2} \sigma^2 \beta^2 \frac{\delta^2 S_n(\lambda\beta)}{\delta\beta^2} \left[M_{n+4}^m\ P_{n+4}^m + M_{n+2}^m\, P_{n+2}^m + M_n^m\, P_n^m \right.$$

$$\left. + M_{n-2}^m\, P_{n-2}^m + M_{n-4}^m\ P_{n-4}^m \right]$$

$$+ \frac{1}{3} e^2\ I_{n+2}^m\ S_{n+2}\ (\lambda\beta)\ P_{n+2}^m$$

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* *See Collected Scientific Papers*, Vol. II, p. 396.

$$+\frac{1}{3}e^2\sigma\, I^m_{n+2}\ \beta\ \frac{\delta S_{n+2}(\lambda\beta)}{\delta\beta}\Big[B^m_{n+4}\ P^m_{n+4}\ +C^m_{n+2}\ P^m_{n+2}\ +D^m_n\ P^m_n\ \Big]$$

$$+\frac{1}{3}e^2 I^m_{n-2}\ S_{n-2}\ (\lambda\beta)\ P^m_{n-2}$$

$$+\frac{1}{3}e^2\sigma\cdot I^m_{n-2}\ \beta\ \frac{\delta S_{n-2}(\lambda\beta)}{\delta\beta}\Big[B^m_n\ P^m_n\ +C^m_{n-2}\ P^m_{n-2}\ +D^m_{n-4}\ P^m_{n-4}\ \Big]$$

$$+\tau\sum_{t=m}^{\infty}{}'\ I'^m_t\ S_t(\lambda\beta)P^m_t.$$

This equation must hold at every point on the boundary surface. Hence equating to zero the co-efficients of the various surface harmonics, we get

$$\tau I'^m_{n+4}\ S_{n+4}\ (\lambda\beta)+\frac{1}{3}e^2\sigma I^m_{n+2}\ B^m_{n+4}\ \beta\frac{\delta S_{n+2}(\lambda\beta)}{\delta\beta}+\frac{1}{2}\sigma^2\beta^2 M^m_{n+4}\ \frac{\delta^2 S_n(\lambda\beta)}{\delta\beta^2}$$

$$+\tau N^m_{n+4}\ \beta\frac{\delta S_n\ (\lambda\beta)}{\delta\beta}=0, \qquad \ldots \qquad \ldots \qquad \ldots\ (1)$$

$$\tau I'^m_{n+2}\ S_{n+2}\ (\lambda\beta)+\frac{1}{3}e^2\sigma\beta\frac{\delta S_{n+2}(\lambda\beta)}{\delta\beta}I^m_{n+2}\ C^m_{n+2}\ +\frac{1}{3}e^2 I^m_{n+2}\ S_{n+2}(\lambda\beta)$$

$$+\frac{1}{2}\sigma^2\beta^2\frac{\delta^2 S_n(\lambda\beta)}{\delta\beta^2}M^m_{n+2}+\tau\beta\frac{\delta S_n(\lambda\beta)}{\delta\beta}\ N^m_{n+2}+\sigma\beta\ \frac{\delta S_n(\lambda\beta)}{\delta\beta}B^m_{n+2}=0, \quad (2)$$

$$S_n(\lambda\beta)+\sigma\beta\ \frac{\delta S_n(\lambda\beta)}{\delta\beta}C^m_n+\tau\beta\frac{\delta S_n(\lambda\beta)}{\delta\beta}\ N^m_n+\frac{1}{2}\sigma^2\beta^2\frac{\delta^2 S_n(\lambda\beta)}{\delta\beta^2}M^m_n$$

$$+\frac{1}{3}e^2\sigma\beta\frac{\delta S_{n+2}(\lambda\beta)}{\delta\beta}I^m_{n+2}\ D^m_n+\frac{1}{3}e^2\sigma\beta\frac{\delta S_{n-2}(\lambda\beta)}{\delta\beta}I^m_{n-2}B^m_n=0, \qquad \ldots\ (3)$$

$$\tau I'^m_{n-2}S_{n-2}(\lambda\beta)+\frac{1}{3}e^2\sigma\beta\frac{\delta S_{n-2}(\lambda\beta)}{\delta\beta}I^m_{n-2}C^m_{n-2}+\frac{1}{3}e^2 S_{n-2}(\lambda\beta)I^m_{n-2}$$

$$+\frac{1}{2}\sigma^2\beta^2\frac{\delta^2 S_n(\lambda\beta)}{\delta\beta^2}\ M^m_{n-2}+\Big[\tau N^m_{n-2}+\sigma D^m_{n-2}\Big]\beta\frac{\delta S_n(\lambda\beta)}{\delta\beta}=0, \qquad \ldots\ (4)$$

$$\tau\ I'^m_{n-4}\ S_{n-4}\ (\lambda\beta)\ +\frac{1}{3}\ e^2\sigma\beta\frac{\delta S_{n-2}(\lambda\beta)}{\delta\beta}\ I^m_{n-2}\ D^m_{n-4}$$

$$+\frac{1}{2}\sigma^2\beta^2\ \frac{\delta^2\ S_n(\lambda\beta)}{\delta\beta^2}\ M^m_{n-4}+\tau\beta\frac{\delta S_n\ (\lambda\beta)}{\delta\beta}N^m_{n-4}=0\,; \qquad \ldots\ (5)$$

all the other I′'s are zero.

Thus the unknown constants are determined and the required expression for W_n^m is given by

$$W_n^m = S_n(\lambda r) P_n^m \cos m\phi + \left[\frac{1}{3} e^2 I_{n+2}^m + \tau I'^m_{n+2}\right] S_{n+2}(\lambda r) P_{n+2}^m$$

$$\times \cos m\phi + \tau I'^m_{n+4} S_{n+4}(\lambda r) P_{n+4}^m \cos m\phi + \tau I'^m_{n-4} S_{n-4}(\lambda r)$$

$$\times P_{n-4}^m \cos m\phi + \left[\frac{1}{3} e^2 I_{n-2}^m + \tau I'^m_{n-2}\right] S_{n-2}(\lambda r) P_{n-2}^m \cos m\phi, \quad (6)$$

where I''s are given by the equations (1), (2), (4) and (5) and λ is a root of the equation (3)

In terms of a and e, the equation (3) becomes

$$S_n(\lambda a) - \frac{1}{3}e^2\left(1 - C_n^m\right) a\frac{\delta S_n}{\delta a} + e^4\left(\frac{1}{21} C_n^m + \frac{3}{35} N_n^m - \frac{2}{15}\right) a \frac{\delta S_n}{\delta a}$$

$$+ \frac{1}{18} e^4 \left(M_n^m - 2 C_n^m + 1\right) a^2 \frac{\delta^2 S_n}{\delta a^2} + \frac{1}{9} e^4 D_n^m I_{n+2}^m a \frac{\delta S_{n+2}}{\delta a}$$

$$+ \frac{1}{9} e^4 B_n^m I_{n-2}^m a \frac{\delta S_{n-2}}{\delta a} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

W_n for ellipsoid with three unequal axes.

13. Let e_1 and e_2 be the eccentricities of the two principal diametral sections of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a > b > c$, by planes passing through the major axis. Then, *neglecting the fourth and higher powers of e_1 and e_2*, the equation to the ellipsoid is

$$r = a\{1 - \frac{1}{6}(e_1^2 + e_2^2)(P_0 - P_2) + \frac{1}{12}(e_2^2 - e_1^2) P_2^2 \cos 2\phi\},$$

i.e., $$r = \gamma(1 + \epsilon_1 P_2 + \epsilon_2 P_2^2 \cos 2\phi),$$

where $$\gamma = a\{1 - \frac{1}{6}(e_1^2 + e_2^2)\},$$

$$\epsilon_1 = \frac{1}{6}(e_1^2 + e_2^2),$$

and $$\epsilon_2 = \frac{1}{12}(e_2^2 - e_1^2).$$

Now assume that

$$W_o = S_o(\lambda r) + \sum_{t=m}^{\infty}{}' \sum_{m=o}^{\infty} R_{m,t}\, S_t(\lambda r)\, P_t^m(\cos\theta)\cos m\phi,$$

where $R_{m,t}$ is an unknown constant to be determined and $\sum\limits_{t=m}^{\infty}{}'$ refers to all values of t from m up to ∞, except $t=0$.

Then it is evident that W_o satisfies the partial differential equation (I).

To satisfy the boundary condition, we must have

$$O = S_o(\lambda\gamma) + \gamma\frac{\delta S_o(\lambda\gamma)}{\delta\gamma}(\epsilon_1 P_2 + \epsilon_2 P_2^2\cos 2\phi)$$

$$+ \sum_{t=m}^{\infty}{}' \sum_{m=o}^{\infty} R_{m,t}\, S_t(\lambda\gamma)\, P_t^m(\cos\theta)\cos m\phi;$$

the unknown constants R's being assumed to be of the same order as ϵ_1 or ϵ_2

Therefore, equating to zero the coefficients of the various surface harmonics, we get

$$S_o(\lambda\gamma) = 0, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$R_{o,2}\, S_2(\lambda\gamma) + \epsilon_1\gamma\frac{\delta S_o(\lambda\gamma)}{\delta\gamma} = 0, \quad \ldots \quad \ldots \quad (2)$$

$$R_{2,2}\, S_2(\lambda\gamma) + \epsilon_2\gamma\frac{\delta S_o(\lambda\gamma)}{\delta\gamma} = 0; \quad \ldots \quad \ldots \quad (3)$$

and all the other R's are zero.

Thus the R's are determined, and we get finally the required expression W_o to be

$$W_o = S_o(\lambda r) - \frac{1}{6}(e_1^2 + e_2^2)\frac{a\frac{\delta S_o(\lambda a)}{\delta a}}{S_2(\lambda a)} S_2(\lambda r)\, P_2(\cos\theta)$$

$$+ \frac{1}{12}(e_1^2 - e_2^2)\frac{a\frac{\delta S_o(\lambda a)}{\delta a}}{S_2(\lambda a)} S_2(\lambda r)\, P_2^2(\cos\theta)\cos 2\phi,$$

where λ is a root of the equation (1).

But the roots of the equation (1) are given by $\lambda\gamma = i\pi$.

Hence, in terms of a and the eccentricities, we have

$$\lambda a = i\pi + \frac{1}{6}(e_1^2 + e_2^2)\, i\pi.$$

14. In order to get a closer approximation to the value of W_o, I will now *retain the fourth power of the eccentricities and neglect the sixth and higher powers*; so that the equation of the ellipsoid may be written as

$$r=c'\,(1+\epsilon'_1 P_2+\epsilon'_2 P_4+\epsilon_3 P_2^2 \cos 2\phi+\epsilon_4 P_4^2 \cos 2\phi+\epsilon_5 P_4^4 \cos 4\phi),$$

where

$$c'=a[1-\frac{1}{6}(e_1^2+e_2^2)-\frac{11}{120}(e_1^4+e_2^4)+\frac{1}{20}e_1^2e_2^2],$$

$$\epsilon_1'=\frac{1}{6}(e_1^2+e_2^2)+\frac{11}{126}(e_1^4+e_2^4)-\frac{1}{63}e_1^2e_2^2,$$

$$\epsilon_2'=\frac{9}{280}(e_1^4+e_2^4)+\frac{3}{140}e_1^2e_2^2,$$

$$\epsilon_3=-\frac{1}{12}(e_1^2-e_2^2)-\frac{11}{252}(e_1^4-e_2^4),$$

$$\epsilon_4=-\frac{1}{280}(e_1^4-e_2^4),$$

and $\epsilon_5=\dfrac{1}{2240}(e_1^2-e_2^2)^2.$

Let us now assume

$$W_o=S_o(\lambda r)+R_2S_2(\lambda r)\,(\sigma_1P_2+\sigma_2P_2^2\cos 2\phi)$$
$$+\sum_{t=m}^{\infty}{}'\sum_{m=o}^{\infty} R'_{m,t}S_t(\lambda r)\,P_t^m\cos m\phi,$$

where $R'_{m,t}$ is an unknown constant *of the order* (*eccentricity*)^4 and $\sum\limits_{t=m}^{\infty}{}'$ refers to all values of t from m up to ∞, except $t=o$; further

$$\sigma_1=\frac{1}{6}(e_1^2+e_2^2),\ \sigma_2=-\frac{1}{12}(e_1^2-e_2^2),\ R_2=-a\,\frac{\delta S_o(\lambda a)}{\delta a}\Big/S_2(\lambda a).$$

Then it is evident W_o satisfies the partial differential equation (I). To satisfy the boundary condition we must have

$$0=S_o(\lambda c')+c'\,\frac{\delta S_o(\lambda c')}{\delta c'}(\epsilon'_1P_2+\epsilon'_2P_4+\epsilon_3P_2^2\cos 2\phi+\epsilon_4P_4^2\cos 2\phi$$
$$+\epsilon_5P_4^4\cos 4\phi)$$

$$+\frac{c'^2}{2!}\frac{\delta^2 S_0(\lambda c')}{\delta c'^2}\ (\epsilon'_1 P_2+\epsilon'_2 P_4+\epsilon_3 P_2^2\cos 2\phi+\epsilon_4 P_4^2\cos 2\phi$$

$$+\epsilon_5 P_4^4\cos 4\phi)^2$$

$$+R_2 S_2(\lambda c')(\sigma_1 P_2+\sigma_2 P_2^2\cos 2\phi)$$

$$+R_2 c'\ \frac{\delta S_2(\lambda c')}{\delta c'}\ (\sigma_1 P_2+\sigma_2 P_2^2\cos 2\phi)(\epsilon'_1 P_2+\epsilon'_2 P_4$$

$$+\epsilon_3 P_2^2\cos 2\phi+\epsilon_4 P_4^2\cos 2\phi+\epsilon_5 P_4^4\cos 4\phi)$$

$$+\sum_{t=m}^{\infty}{}'\ \sum_{m=o}^{\infty}\ R'_{m,t}\ S_t\ (\lambda c')P_t^m\cos m\phi,$$

since e_1^6, e_2^6 and higher powers of e_1 and e_2 are neglected.

Again

$$P_2\ \times\ P_2^2=\frac{3}{35}\ P_4^2-\frac{2}{7}\ P_2^2,$$

$$P_2^2\ \times\ P_2^2=\frac{3}{35}\ P_4^4$$

$$=24\ [\frac{1}{5}-\frac{2}{7}\ P_2\ +\frac{3}{35}\ P_4],$$

Hence

$$0=S_0(\lambda c')+c'\frac{\delta S_0(\lambda c')}{\delta c'}\ [\epsilon'_1 P_2+\epsilon'_2 P_4+\epsilon_3 P_2^2\cos 2\phi+\epsilon_4 P_4^2\cos 2\phi$$

$$+\epsilon_5 P_4^4\cos 4\phi]$$

$$+\ \frac{1}{2}\sigma_1{}^2 c'^2\ \frac{\delta^2 S_0(\lambda c')}{\delta c'^2}\left[\frac{18}{35}\ P_4+\ \frac{2}{7}P_2+\ \frac{1}{5}\right]$$

$$+\frac{3}{140}\ \sigma_2{}^2\ \ c'^2\ \ \frac{\delta^2 S_0(\lambda c')}{\delta c'^2}P_4^4\cos 4\phi$$

$$+6\sigma_2{}^2 c'^2\ \ \frac{\delta^2 S_0(\lambda c')}{\delta c'^2}\left[\frac{3}{35}P_4-\frac{2}{7}\ P_2\ +\ \frac{1}{5}\right]$$

$$+\sigma_1\sigma_2 c'^2\ \frac{\delta^2 S_0(\lambda c')}{\delta c'^2}\left[\frac{3}{35}\ P_4^2\ -\ \frac{2}{7}\ P_2^2\ \right]\cos 2\phi$$

$$+R_2 S_2\ (\lambda c')\ [\sigma_1 P_2+\sigma_2\ P_2^2\cos 2\phi]$$

$$+\sigma_1{}^2 R_2 c'\ \ \frac{\delta S_2(\lambda c')}{\delta c'}\ \left[\frac{18}{35}\ P_4+\frac{2}{7}\ P_2+\frac{1}{5}\right]$$

$$+2\sigma_2\sigma_1 \ R_2 c' \frac{\delta S_2(\lambda c')}{\delta c'} \left[\frac{3}{35} \ P_4^2 - \frac{2}{7} P_2^2 \right] \cos 2\phi$$

$$+\frac{3}{70}\sigma_2^2 R_2 c' \frac{\delta S_2(\lambda c')}{\delta c'} P_4^4 \cos 4\phi$$

$$+12\ \sigma_2^2 R_2 c' \frac{\delta S_2(\lambda c')}{\delta c'} \left[\frac{3}{35} \ P_4 - \frac{2}{7} P_2 + \frac{1}{5} \right]$$

$$+\sum_{t=m}^{\infty} \sum_{m=o}^{\infty} R'_{m,t} S_t \ (\lambda c') P_t^m \cos m\phi.$$

This must hold at all points on the surface. Hence equating to zero the co-efficients of the various surface harmonics, we get

$$S_o(\lambda c') + \frac{1}{5}(\sigma_1^2 + 12\ \sigma_2^2) R_2 \ c' \ \frac{\delta S_2(\lambda c')}{\delta c'}$$

$$+\left(\frac{1}{10}\sigma_1^2 + \frac{6}{5}\sigma_2^2\right) c'^2 \ \frac{\delta^2 S_0(\lambda c')}{\delta c'^2} = 0, \quad \ldots \quad \ldots \quad (1)$$

$$R'_{0,2} \ S_2 \ (\lambda c') + \epsilon'_1 \ c' \ \frac{\delta S_0(\lambda c')}{\delta c'} + \left(\frac{1}{7}\sigma_1^2 - \frac{12}{7}\sigma_2^2\right) c'^2 \frac{\delta^2 S_0(\lambda c')}{\delta c'^2}$$

$$+\sigma_1 R_2 \ S_2(\lambda c') + \frac{2}{7}(\sigma_1^2 - 12\sigma_2^2) R_2 \ c' \frac{\delta S_2 \ \lambda c')}{\delta c'} = 0, \quad \ldots \quad (2)$$

$$R'_{0,4} \ S_4 \ (\lambda c') + \epsilon'_2 \ c' \frac{\delta S_0(\lambda c')}{\delta c'} + \left(\frac{9}{35}\sigma_1^2 + \frac{18}{35}\sigma_2^2\right) c'^2 \frac{\delta^2 S_0(\lambda c')}{\delta c'^2}$$

$$+\left(\frac{18}{35}\sigma_1^2 + \frac{36}{35}\sigma_2^2\right) R_2 \ c' \frac{\delta S_2(\lambda c')}{\delta c'} = 0, \quad \ldots \quad \ldots \quad (3)$$

$$R'_{2,2} \ S_2 \ (\lambda c') + \epsilon_3 c' \frac{\delta S_0(\lambda c')}{\delta c'} - \frac{2}{7}\sigma_1\sigma_2 \ c'^2 \frac{\delta^2 S_0(\lambda c')}{\delta c'^2}$$

$$+\sigma_2 R_2 S_2(\lambda c') - \frac{4}{7}\sigma_1\sigma_2 \ R_2 c' \frac{\delta S_2(\lambda c')}{\delta c'} = 0, \ \ldots \quad \ldots \quad (4)$$

$$R'_{2,4} \ S_4 \ (\lambda c') + \epsilon_4 c' \ \frac{\delta S_0(\lambda c')}{\delta c'} + \frac{3}{35}\sigma_1\sigma_2 \ c'^2 \frac{\delta^2 S_0(\lambda c')}{\delta c'^2}$$

$$+\frac{6}{35}\sigma_1 \ \sigma_2 \ R_2 c' \frac{\delta S_2(\lambda c')}{\delta c'} = 0, \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$R'_{4,4} \ S_4(\lambda c') + \epsilon_5 \ c' \ \frac{\delta S_0(\lambda c')}{\delta c'} + \frac{3}{140}\sigma_2^2 \ c'^2 \frac{\delta^2 S_0(\lambda c')}{\delta c'^2}$$

$$+\frac{3}{70}\sigma_2^2 R_2 c' \frac{\delta S_2(\lambda c')}{\delta c'} = 0, \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

and all the other R′'s are zero.

Therefore the required expression for W_0 is

$$W_0 = S_0(\lambda r)+(R_2\sigma_1+R'_{0,2})\,S_2(\lambda r)\,P_2+R'_{0,4}\,S_4(\lambda r)\,P_4$$
$$+(R_2\sigma_2+R'_{2,2})\,S_2(\lambda r)\,P_2^2\cos 2\phi$$
$$+R'_{2,4}\,S_4(\lambda r)\,P_4^2\cos 2\phi$$
$$+R'_{4,4}\,S_4(\lambda r)\,P_4^4\cos 4\phi,$$

where $R'_{0,2}$ etc., are determined from the equations (2) — (6) and λ is a root of the equation (1).

15. I will now proceed to the solution of the equation (1) of the preceding article. When the values of R_2, c', σ_1, etc., in terms of a, e_1 and e_2 have been substituted, the equation becomes

$$S_0(\lambda a) - a\frac{\delta S_0(\lambda a)}{\delta a}\left\{\frac{1}{6}(e_1^2+e_2^2)+\frac{11}{120}(e_1^4+e_2^4)-\frac{1}{20}e_1^2e_2^2\right\}$$
$$+a^2\frac{\delta^2 S_0(\lambda a)}{\delta a^2}\left\{\frac{1}{60}(e_1^2+e_2^2)^2+\frac{1}{120}(e_1^2-e_2^2)^2\right\}$$
$$-\left\{\frac{1}{180}(e_1^2+e_2^2)^2+\frac{1}{60}(e_1^2-e_2^2)^2\right\}a\frac{\delta S_0}{\delta a}.\,a\frac{\delta S_2}{\delta a}\Big/S_2$$
$$=0. \qquad \ldots \quad (7)$$

If we neglect eccentricity altogether, the equation reduces to $S_0(\lambda a)=0$, whose roots are given by $\lambda a=i\pi$. Therefore let the full value of λa be

$$i\pi+l_1e_1^2+l_2e_2^2+l_3e_1^4+l_4e_2^4+l_5e_1^2e_2^2+\ldots\ldots,$$

l_1, l_2 etc...... being unknown constants to be determined. It is clear from the nature of the equation (7) that λa will not have any terms containing e_1e_2, $e_1^3e_2$ etc.

Then it is easy to calculate the following:—

$$S_0(\lambda a)=\left(l_1e_1^2+l_2e_2^2+l_3e_1^4+l_4e_2^4+l_5e_1^2e_2^2\right)\frac{\cos i\pi}{i\pi}$$
$$-\left(l_1^2e_1^4+l_2^2e_2^4+2l_1l_2e_1^2e_2^2\right)\frac{\cos i\pi}{i^2\pi^2}+\ldots$$

$$a\frac{\delta S_0(\lambda a)}{\delta a}=\cos i\pi-\left(l_1e_1^2+l_2e_2^2+l_3e_1^4+l_4e_2^4+l_5e_1^2e_2^2\right)\frac{\cos i\pi}{i\pi}+\ldots$$

$$a^2\frac{\delta^2 S_0(\lambda a)}{\delta a^2}=-2\cos i\pi+\ldots\ldots$$

$$a\frac{\delta S_0(\lambda a)}{\delta a}\,a\frac{\delta S_2(\lambda a)}{\delta a}\Big/S_2(\lambda a)=\frac{1}{3}(i^2\pi^2-9)\cos i\pi+\ldots$$

Therefore substituting the above expressions for $S_n(\lambda a)$, etc. ... in the equation (7), and equating to zero the co-efficients of e_1^2 etc., ... we obtain finally

$$\lambda a = i\pi + \frac{i\pi}{6}\left(e_1^2 + e_2^2\right) + i\pi\left(\frac{3}{40} + \frac{i^2\pi^2}{135}\right)\left(e_1^4 + e_2^4\right)$$

$$+ i\pi\left(\frac{1}{20} - \frac{i^2\pi^2}{135}\right)e_1^2 e_2^2 \quad \ldots$$

W_n^m for ellipsoid with three unequal axes.

16. We shall *neglect the fourth and higher powers of the eccentricities* so that the equation of the ellipsoid will be the same as in Art. 13, *viz.*,

$$r = \gamma\left(1 + \epsilon_1 P_2 + \epsilon_2 P_2^2 \cos 2\phi\right).$$

Let us assume

$$W_n^m = S_n(\lambda r) P_n^m(\cos\theta)\cos m\phi + \sum_{t=p}^{\infty}{}' \sum_{p=0}^{\infty} T_{t,p} P_t^p(\cos\theta)\cos p\phi,$$

where $T_{t,p}$ is an unknown constant *of the order (eccentricity)*^{2} and $\sum_{t=p}^{\infty}{}'$ refers to all values of t from p to ∞, except $t=n$. Then W_n^m satisfies the differential equation (I). To satisfy the boundary condition we must have

$$0 = S_n(\lambda\gamma) P_n^m \cos m\phi$$

$$+ \gamma \frac{\delta S_n(\lambda\gamma)}{\delta\gamma}\left(\epsilon_1 P_2 + \epsilon_2 P_2^2 \cos 2\phi\right) P_n^m \cos m\phi$$

$$+ \sum_{t=p}^{\infty}{}' \sum_{p=o}^{\infty} T_{t,p}\; S_t(\lambda\gamma) P_t^p \cos p\phi.$$

Now

$$\cos 2\phi \cos m\phi = \frac{1}{2}\cos(m+2)\phi + \frac{1}{2}\cos(m-2)\phi,$$

$$P_2^2 \times P_n^m = B_{n+2}^{m+2} P_{n+2}^{m+2} - C_n^{m+2} P_n^{m+2} + D_{n-2}^{m+2} P_{n-2}^{m+2},$$

and $P_2^2 \times P_n^m = B'^{m-2}_{n+2} P_{n+2}^{m-2} + C'^{m-2}_n P_n^{m-2} + D'^{m-2}_{n-2} P_{n-2}^{m-2},$

where

$$\overline{B}^{m+2}_{n+2}=\frac{3}{(2n+3)(2n+1)},$$

$$\overline{C}^{m+2}_{n}=\frac{6}{(2n+3)(2n-1)},$$

$$\overline{D}^{m+2}_{n-2}=\frac{3}{(2n+1)(2n-1)},$$

$$B'^{m-2}_{n+2}=3(n-m+1)(n-m+2)(n-m+3)(n-m+4)\Big/(2n+3)(2n+1),$$

$$C'^{m-2}_{n}=-6(n+m)(n+m-1)(n-m+1)(n-m+2)\Big/(2n+3)(2n-1),$$

$$D'^{m-2}_{n-2}=3(n+m)(n+m-1)(n+m-2)(n+m-3)\Big/(2n+1)(2n-1).$$

Hence on the boundary we must have

$$0=S_n(\lambda\gamma)P^m_n\cos m\phi+\epsilon_1\gamma\frac{\delta S_n(\lambda\gamma)}{\delta\gamma}\Big[B^m_{n+2}P^m_{n+2}+C^m_n P^m_n+D^m_{n-2}P^m_{n-2}\Big]\cos m\phi$$

$$+\frac{1}{2}\epsilon_2\gamma\frac{\delta S_n(\lambda\gamma)}{\delta\gamma}\Big[\overline{B}^{m+2}_{n+2}P^{m+2}_{n+2}-\overline{C}^{m+2}_n P^{m+2}_n+\overline{D}^{m+2}_{n-2}P^{m+2}_{n-2}\Big]\cos(m+2)\phi$$

$$+\frac{1}{2}\epsilon_2\gamma\frac{\delta S_n(\lambda\gamma)}{\delta\gamma}\Big[B'^{m-2}_{n+2}P^{m-2}_{n+2}+C'^{m-2}_n P^{m-2}_n+D'^{m-2}_{n-2}P^{m-2}_{n-2}\Big]\cos(m-2)\phi$$

$$+\sum_{t=p}^{\infty}{}'\sum_{p=0}^{\infty}T_{t,p}\,S_t(\lambda\gamma)\,P^p_t\cos p\phi.$$

This must hold at every point on the surface. Hence equating to zero the co-efficients of the various surface harmonics, we get

$$S_n(\lambda\gamma)+\epsilon_1\gamma\frac{\delta S_n(\lambda\gamma)}{\delta\gamma}C^m_n=0,\qquad\ldots\qquad\ldots\qquad\ldots\qquad(1)$$

$$T_{n+2,m}S_{n+2}(\lambda\gamma)+\epsilon_1\gamma\frac{\delta S_n(\lambda\gamma)}{\delta\gamma}B^m_{n+2}=o,\qquad\ldots\qquad(2)$$

$$T_{n-2,m}S_{n-2}(\lambda\gamma)+\epsilon_1\gamma\frac{\delta S_n(\lambda\gamma)}{\delta\gamma}D^m_{n-2}=o,\qquad\ldots\qquad(3)$$

$$T_{n+2,m+2}S_{n+2}(\lambda\gamma)+\frac{1}{2}\epsilon_2\gamma\frac{\delta S_n(\lambda\gamma)}{\delta\gamma}\overline{B}^{m+2}_{n+2}=o,\qquad\ldots\qquad(4)$$

$$T_{n,m+2}S_n(\lambda\gamma)-\frac{1}{2}\epsilon_2\gamma\frac{\delta S_n(\lambda\gamma)}{\delta\gamma}\overline{C}^{m+2}_n=o,\qquad\ldots\qquad(5)$$

$$T_{n-2,\,m+2}\, S_{n-2}\,(\lambda\,\gamma) + \frac{1}{2}\epsilon_2\,\gamma\,\frac{\delta\, S_n\,(\lambda\,\gamma)}{\delta\,\gamma}\, D^{m+2}_{n-2} = o, \quad \ldots \quad (6)$$

$$T_{n+2,\,m-2}\, S_{n+2}\,(\lambda\,\gamma) + \frac{1}{2}\epsilon_2\,\gamma\,\frac{\delta\, S_n\,(\lambda\,\gamma)}{\delta\,\gamma}\, B'^{m-2}_{n+2} = o, \quad \ldots \quad (7)$$

$$T_{n,\,m-2}\, S_n\,(\lambda\,\gamma) + \frac{1}{2}\epsilon_2\,\gamma\,\frac{\delta\, S_n\,(\lambda\,\gamma)}{\delta\,\gamma}\, C'^{m-2}_{n} = o, \quad \ldots \quad (8)$$

$$T_{n-2,\,m-2}\, S_{n-2}\,(\lambda\,\gamma) + \frac{1}{2}\epsilon_2\,\gamma\,\frac{\delta\, S_n\,(\lambda\,\gamma)}{\delta\,\gamma}\, D'^{m-2}_{n-2} = o, \quad \ldots \quad (9)$$

and all the other T's are zero.

Thus the unknown constants are determined, and the required expression for W^m_n is

$$\begin{aligned} W^m_n = \Big[& S_n\,(\lambda r)\, P^m_n + T_{n+2,\,m}\, S_{n+2}\,(\lambda\, r)\, P^m_{n+2} + T_{n-2,\,m}\, S_{n-2}\,(\lambda\, r) \\ & \times P^m_{n-2}\Big] \cos m\,\phi + \Big[T_{n+2,\,m+2}\, S_{n+2}\,(\lambda\, r)\, P^{m+2}_{n+2} + T_{n,\,m+2}\, S_n\,(\lambda\, r) \\ & \times P^{m+2}_{n} + T_{n-2,\,m+2}\, S_{n-2}\,(\lambda\, r)\, P^{m+2}_{n-2}\Big] \cos\,(m + 2)\,\phi \\ & + \Big[\, T_{n+2,\,m-2}\, S_{n+2}\,(\lambda\, r)\, P^{m-2}_{n+2} + T_{n,\,m-2}\, S_n\,(\lambda\, r)\, P^{m-2}_{n} \\ & + T_{n-2,\,m-2}\, S_{n-2}\,(\lambda r)\, P^{m-2}_{n-2}\Big] \cos\,(m - 2)\,\phi, \end{aligned}$$

where T's are given by the equations (2) — (9) and λ is a root of the equation (1).

But, expressed in terms of a and the eccentricities, this equation becomes

$$S_n\,(\lambda a) - \frac{1}{2}\cdot\frac{(n^2+n-1)+m^2}{(2n+3)\,(2n-1)}\quad (e_1^{\,2} + e_2^{\,2})\; a\frac{\delta S_n\,(\lambda a)}{\delta a} = o. \quad \ldots \quad (10)$$

Therefore, we obtain

$$\lambda a \;= k\;\left\{1 + \frac{1}{2}\cdot\frac{(n^2+n-1)+m^2}{(2n+3)\,(2n-1)}\;(e_1^2 + e_2^2)\right\},$$

corresponding to the root k of $S_n\,(\lambda a) = o$.

Conclusion.

17. In the preceding articles, I have studied the normal function of the type W_n^m corresponding to

$$S_n(\lambda r) P_n^m(\cos\theta)\cos m\phi,$$

the normal function for the sphere. It is obvious that the normal function $\overline{W}_n^m$ corresponding to

$$S_n(\lambda r) P_n^m(\cos\theta)\sin m\phi$$

may be investigated in a similar way.

Review

Lectures on ten British mathematicians of the Nineteenth Century, by Alexander Macfarlane. pp. 148 (New York: John Wiley & Sons, 1916). Price 5s. 6d. net.

During the years 1901-1904, the late Dr. Alexander Macfarlane delivered at Lehigh University, lectures on Twenty-five British Mathematicians of the Nineteenth Century. The manuscripts of twenty of these lectures were found by the editors to be almost ready for the press, although some marginal notes by the author indicated that he had certain additions in view. These notes have, however, been disregarded and the editors present in this book ten lectures on ten Pure Mathematicians in their original form, hoping to issue in a future volume, the other lectures which are on British Mathematicians whose main work lay in the domains of Physics and Astronomy. The book opens with a group of portraits of the mathematicians and a brief necrology of the author who himself was no mean mathematician and whose personal acquaintance with British mathematicians of the Nineteenth Century imparts a personal touch to the lectures which greatly adds to their interest.

The lectures have been published as No. 17 of the well-known American Mathematical Monographs edited by Messrs. Merriman & Woodward. Their venture is indeed welcome because the times are such that the world requires to be reminded that Great Britain has been the mother of men of genius and culture whose achievements have not only shed lustre on their land of birth, but have also been epoch-making and path-breaking for the rest of the world. And by depicting the life and work of ten *Pure* mathematicians—men whose work lay entirely in the domain of *pure thought*, irrespective of *applications*, Dr. Macfarlane has skilfully illuminated that particular side of mathematical culture which is likely to be lost sight of at the present day when the triumphs of Applied Science are so prominent in Arts, in Commerce and in War.

The selection has been truly representative. Beginning with George Peacock (1791-1858), the author places before his reader the

memorable work done by Augustus De Morgan (1806-1871), Sir William Rowan Hamilton (1805-1865), George Boole (1815-1864), Arthur Cayley (1821-1895), William Kingdom Clifford (1845-1879), Henry John Stephen Smith (1826-1883), James Joseph Sylvester (1814-1897), Thomas Penyngton Kirkman (1806-1895), and, Isaac Todhunter (1820-1884).

To appreciate rightly the place assigned to George Peacock in the roll of British Mathematicians, it is necessary to turn for a moment to the state of Mathematical Science in Great Britain at the beginning of the Nineteenth Century. In this respect Dr. Macfarlane has mentioned Jacobi's saying when he visited Cambridge in 1842. When dining as a guest at the table of one of the colleges, Jacobi, being asked who in his opinion was the greatest of the living mathematicians of England, replied "there is none." This remark, although not correct at the time it was made, was correct as representing the state of the Science at the beginning of the 19th Century. "In England the Sceptre of Mathematics passed from the hands of Wallis (1616-1703) to the man who was to raise the exact sciences to a height hitherto unattained—Isaac Newton. The profoundly beneficient influence of Newton on his time may be gauged by the output of those who openly announced themselves his disciples and followers." But the direction given to their investigations (especially by the work of Maclaurin undertaken with the firm conviction that the best path to follow was that opened by Newton) had a deplorable effect. It completely isolated England from the enthusiastic and productive movement taking place on the continent and, as Dr. Macfarlane has pointed out, "the University of Cambridge—the centre of mathematical culture in England—had settled down to "the Study of Newton instead of Nature" and they had followed him in one grand mistake—the ignoring of the differential notation in the calculus. Dr. Macfarlane mentions that the first British Mathematician who succeeded in liberating himself from the prejudices of nationality and in throwing off the shackles of the Newtonian notation and methods, was James Ivory (1765-1842). There were others to follow and the first quarter of the 19th century witnessed the foundation at Cambridge (in 1815) of the famous Analytical Society by "a group of daring young mathematicians determined to substitute the policy of *Ententes cordiales* for that of *splendid isolation.*" Great was the influence of George Peacock

who, with Babbage and Herschel, was the life and soul of the Analytical Society, in furthering its objects. Dr. Macfarlane has devoted a considerable portion of his lecture to describing how the persistent efforts of the members of the Society succeeded in breaking down the prejudices and in introducing the continental methods into the Cambridge School. This was the greatest service which George Peacock rendered to the progress of the mathematical science in England.

The remainder of the lecture is devoted to an account of the other results obtained by the reforming energy of George Peacock, *viz.*, the foundation in collaboration with Babbage and Herschel, of the Astronomical Society of London, the Astronomical Observatory at Cambridge and the Philosophical Society of Cambridge; and also to an account of Peacock's contributions to mathematical analysis with the object of placing Algebra on a strictly logical basis. Although Peacock's well-known principle "of permanence of equivalent forms" is rightly said to be now "as dead as a doornail," Dr. Macfarlane has given sufficient materials to show that Peacock was one of the few, who realized Algebra as "an abstract symbolism with more or less arbitrary fundamental rules." In this sense he may be said to be the forerunner of the later Symbolical school of which the chief exponents were George Boole, Augustus De Morgan and D. F. Gregory.

Dr. Macfarlane's lecture on Augustus De Morgan is highly interesting to those who have not followed that unique personality in his numerous writings on the Philosophy of Mathematics and in his contributions to the Athenæum which were collected and published afterwards by his wife as the Budget of Paradoxes, a most delightful and erudite miscellany, displaying De Morgan's wide reading and researches, his quaint humour and his views on Circle Squarers, Angle-trisectors, Cyclometrical Paradoxers, Duplicators of the cube, Constructors of perpetual motion, Subverters of gravitation and on other things, Literary, Scientific and Social. De Morgan's work rendered as the first Professor of Mathematics in the newly founded London University, his contributions to the Society for the diffusion of useful knowledge, his enthusiastic work as the first President of the Mathematical Society in London, founded at the suggestion of his son, George, who had already, before his untimely death, acquired some distinction in Mathematics,—De Morgan

loved to hear him called as the "younger Bernoulli"—his passages-at-arms in the discussion on the doctrine of the quantification of the predicate with Sir William Hamilton of Edinburgh, his highly interesting correspondence with Sir William Rowan Hamilton of Dublin, his remarkable dislike for titles and honours, his contributions on the foundations of Algebra and the last but not least, his Budget of Paradoxes, have all been summed up well by Dr. Macfarlane in the space of about 33 pages. We miss, however, one aspect of De Morgan's character as a mathematician and that aspect, a very prominent one. As has been well said "De Morgan created no branches of knowledge and discovered little of note, yet when the scientific History of England in the 19th Century is written, his name will occupy a prominent position, for he profoundly influenced the opinions of the ordinary man of science and mathematician of his time." What then was the secret of his undoubted power in the mathematical world?

The secret is revealed in "his historical papers and reviews, his occasional lectures on general subjects and in the universal recognition of his desire for *justice* and *scorn of all pretence*." His celebrated "Essays on the life and work of Newton" republished in 1914 by P. E. B. Jourdain, with notes and appendices should be read by all who are interested in the Newton-Leibniz controversy.

Dr. Macfarlane, a pupil of Tait, was an ardent quaternionist and was afterwards the President of the International Association for promoting the study of quaternions. His lecture on Rowan Hamilton is, therefore, eminently well informed and appreciative.

William Rowan Hamilton has been rightly called the Irish Lagrange. And Hamilton himself while studying Lagrange's Mecanique Analytique, called it "a scientific poem." Hamilton's fame as one of the greatest mathematicians of the 19th Century rests, in the opinion of competent authorities, more on his enduring work in Dynamics, than on his discovery of the calculus of quaternions. Dr. Macfarlane's reference to Hamilton's work in Dynamics, however, is meagre in comparison with the account he has given of the discovery of the calculus of quaternions. The materials are mostly taken from Graves's three-volume Life of Sir William Rowan Hamilton—one of the few excellent biographies of scientific men which have yet been written. Dr. Macfarlane has not neglected to duly present the poetic side of Hamilton's character and has quoted at length his poem on

"College ambition" as a fair specimen of his poetical attainments. The poet, the philosopher and the mathematician were all three combined in Hamilton—a fact which Macfarlane's lecture succeeds in bringing home to his readers.

The very interesting third lecture closes with Hamilton's own estimate of himself—"I have very long admired" wrote Hamilton "Ptolemy's description of his great astronomical master, Hipparchus, as a labour-loving and truth-loving man. Be such my epitaph." May such be the epitaph of every cultivator of the fertile fields of science!

A most remarkable career of a self-taught mathematician whose name will be honoured wherever the English language is spoken, is described by Dr. Macfarlane in his fourth lecture. This lecture is valuable as giving a good account of Boole's pioneer work as the inventor of symbolical logic. But Boole's mathematical work has not been sufficiently noticed.

Boole's purely mathematical work—his original researches in the Theory of Linear transformations—the source from which George Salmon derived "his first clear ideas of the nature and objects of the Theory of Linear transformations" and his "general method in analysis," giving a powerful instrument for the integration of differential equations, both total and partial, which brought Boole a gold medal from the Royal Society—his Memoir on Probabilities which won him the Keith prize, his earliest paper on the Calculus of Variations—the result of his studies in Lagrange, his profound papers on Discontinuous functions and Definite Integrals in the Transactions of the Royal Irish Academy, his theorem on "a general transformation of any quantitative function"—a theorem of which Lagrange and Laplace's celebrated theorems are only particular forms, and his other contributions which enriched the Cambridge and Cambridge and Dublin, Mathematical Journals and the Philosophical Magazine, placed at once in the first rank of mathematicians in England, the once humble teacher in a rudimentary school at Waddington who belonging to no University raised the reputation of England by his far-reaching researches in mathematics!

The fifth lecture on Arthur Cayley—"whose soul, too large for vulgar space, in n-dimensions flourished unrestricted" is rather disappointing, in that much of it is devoted to Cayley's views on the utility of quaternions and his well-known controversies with Tait on

the subject. Cayley's great work in Pure Mathematics was the brightest gem in the diadem of scientific England in the 19th century and he, with Hamilton and Sylvester, did epoch-making work in pulling English Mathematics out of the Slough of Despond into which it had fallen in the beginning of the 19th century. He was the founder of what is called "sometimes Modern Algebra, sometimes invariants and covariants, sometimes theory of forms." In this department his work was intertwined with that of Sylvester—his lifelong friend and fellow-worker. Sylvester, then an actuary resident in London, and Cayley, a barrister, used to "walk together round the Courts of Lincoln's Inn, discussing the theory of invariants and covariants." The theory of invariants according to Macmahon, "sprang into existence under the strong hand of Cayley but that it emerged finally as a complete work of art for the admiration of future generations of mathematicians, was largely owing to the flashes of inspiration with which Sylvester's intellect illuminated it." Prof. Gino Loria says "such work as that of Sylvester and Cayley in the theory of Algebraic forms found devotees of the first rank in Hermite, Aronhold, Brioschi, Clebsch. Gordan and finally David Hilbert who placed upon it a worthy crown." Far-reaching has thus been the influence of the work of Cayley and Sylvester in this fascinating branch of Mathematics! Marvellous are Cayley's ten "memoirs on quantics" and the gem appears in the Sixth Memoir in the introduction of the notion of the Absolute into geometry. We find no mention of this in Dr. Macfarlane's Lecture.

Cayley's celebrated Address before the British Association on the progress of Pure Mathematics receives suitable notice and Cayley's views on the foundations of exact science and the philosophy which commended itself to his mind are given by suitable excerpts from that Address. The lecture ends with Cayley's ideas on the Imaginary in Mathematics. Cayley asks "what is an imaginary point? Is there in a plane a point, the co-ordinates of which have given imaginary values? Cayley seems to reply "No" and to fall back on the notion of an imaginary space as the *locus in quo* of the imaginary point." Von Staudt's conception of an imaginary point as an actual geometric-entity does not appear to have been appreciated by Cayley.

But Arthur Cayley was "more than a mathematician." "With a singleness of aim, which Wordsworth could have chosen for his

"Happy Warrior" he persevered to the last in his nobly lived ideal. His life had a significant influence on those who knew him: they admired his character as much as they respected his genius: and they felt that, at his death, a great man had passed from the world".

In his lecture on William Kingdon Clifford, Dr. Macfarlane has chosen to give us only a portion of one of the most gifted sons of England, whose liberality of mind and freedom from bias of any kind have been the watchword among a generation of thinkers. We find in the lecture more of Clifford as the brilliant essayist, the keen and fearless critic and the far-sighted philosopher, than of Clifford as the brilliant mathematician whose writings, like Riemann's, though few, have the indelible stamp on them of the highest order of intellect.

Dr. Mafcarlane's lecture is practically a review of Clifford's Essays and presents only one side—the metaphysical side—of his character, although that side was not unimportant. To those who are interested to see the other side—the mathematical side—would do well to read the masterly Introduction to Clifford's collected mathematical papers by Professor John Stephen Smith,—one of the greatest Mathematicians of the 19th Century. He will find that "Clifford's predilection for geometry lay deep". To Geometry he attributed the widest imaginable scope and at times regarded it as co-extensive with *the whole domain of nature*. "Clifford was above all and before all a geometer. Nine-tenths of his collected papers are geometrical. It is true that in the treatment of geometrical questions he shows a marked preference for symbolical methods and as might be expected, a marvellous command over analytical expression. But, whatever the method employed—and in variety of method Clifford takes an evident pleasure—the properties of space remain the *perpetual subject of his contemplation*." Dr. Macfarlane has appropriately given an instance of this in citing Clifford's views on the foundations of geometry as disclosed in his Address before the British Association in 1872 at Brighton, entitled "The aims and instruments of scientific thought." In this address, Clifford criticised Kant's views on space, by citing the case of the Star-triangle and distinguishing between *exactness* and *universality*. Macfarlane succeeds in controverting Clifford's views on the basis of Beltrami's researches in Non-Euclidean Geometry.

But philosophy apart, Clifford's contributions to Mathematics were real and remarkable. His early paper "on some porismatic problems" relating to Poncelet's Theorem about in-and-circumscribed

polygons led to the discovery of a geometrical theory of the transformation of elliptic functions (No. XXII in his Collected mathematical papers). Attracted by the Abelian integrals in their relation to the Geometry of Situation, he wrote his memoir on the "Canonical Dissection of a Riemann surface." From the study of Abelian integrals he proceeded to that of Theta functions and his memoirs on Double and Multiple Theta functions are the fruits of such study. He rendered great service to English readers by translating early Riemann's celebrated discourse on "the hypotheses which lie at the basis of Geometry," the ideas set forth in which became, it is said, "a part of his intellectual nature." Following Riemann, Clifford held that "the essential properties of space have to be regarded as things still unknown, which we may one day hope to find out by closer observation and more patient reflection and not as axioms to be accepted on the authority of universal experience or of the inner conscience"—a view which will never appeal to the followers of Kant!

Clifford wrote little but what he wrote was pure gold. Dr. Macfarlane has given us Clifford, the mathematico-metaphysician but we have to turn elsewhere to find Clifford as one of the most brilliant mathematicians of all time And by the life he lived, brief though it was, "he fulfilled well and truly the great saying of Spinoza, often in his mind and on his lips: *Homo liberde nulla reminus quam de morte cogitat.*

Dr. Macfarlane's lecture on Henry John Stephen Smith has only to be perused to be appreciated. He brings out most clearly the distinctive features of this great Mathematician's career and his remarkable opinions on men and manners. He gracefully points out that "though addicted to the Theory of numbers, he was not in any sense, a recluse; on the contrary he entered with zest into every form of social enjoyment in Oxford. He had the rare power of utilizing stray hours of leisure and it was in such odd times that he accomplished most of his scientific work. After attending a picnic in the afternoon he could mount to those serene heights in the Theory of numbers,

"Where never creeps a cloud or moves a wind,
Nor ever falls the least white star of snow,
Nor ever lowest roll of thunder moans,
Nor sound of human sorrow mounts, to mar
Their sacred everlasting calm."

"Then he could of a sudden come down from these heights to attend a dinner and could conduct himself there, not as a mathematical genius lost in reverie and pointed out as a poor and eccentric mortal but on the contrary as a thorough man of the world greatly liked by every body." Truly it has been said, Smith was the admirable Chrichton among British mathematicians!

Henry John Stephen Smith's name will always be associated with "the Theory of numbers, the Theory of Elliptic functions and certain new processes of Geometry". When barely 27, he commenced the study of the Higher Arithmetic—the "queen of mathematics" which engaged his undivided attention for years and was never afterwards quite absent from his thoughts. His celebrated Reports on the Theory of numbers, contributed to the British Association in 1859 to 1863 and 1865, bear eloquent testimony to his immense research and are "models of clear exposition and systematic arrangement". They are not merely reports of previous work done by others but are interlaced with original contributions detectable only by those well-versed in the history of the subject. Dr. Macfarlane has closed his lecture appropriately by briefly telling the story of the award of the *Grand prix des sciences mathematiques* of 1882, both to Minkowski and Smith in 1883, after the latter had passed away. The subject was the decomposition of a number into five squares. Professor Smith had, fifteen years before, given the complete theorems not only for five squares but also for seven, in the Royal Society's Proceedings, vol. XVI, pp. 207-208. Although he had not given his *demonstrations*, he had described the *general* Theory, from which these theorems were corollaries. Smith wrote, on the advice of Glaisher, to Charles Hermite, one of the commissioners for the award of the Prize, referring to his previous writings in the Royal Society's proceedings. Hermite replied that Prof. Smith should bring his *demonstrations* before the Paris Academy in the form of a memoir. This he did, while confined to his sofa from the effects of a riding accident and in the midst of his work on Theta and Omega functions. By an unfortunate forgetfulness on the part of Hermite the Report of the Commissioners while awarding the prize omitted all mention of the previous results published by Smith !!

Dr. Macfarlane's eighth lecture presents Sylvester in some of his distinguishing and highly interesting features. There is an

instructive comparison between him and Cayley and towards the end there is another comparison between him and H. J. S. Smith.

Cayley and Sylvester! the pride of England in the 19th century —two great names to conjure with! To Cayley "mathematics was a tract of beautiful country seen at first in the distance, but which will bear to be rambled through and studied in every detail of hill-side and valley, stream, rock, wood and flower." To him and to Sylvester Pure Mathematics was an opportunity for unceasing exploration; or in an other figure, a challenge to carve from the rough block a face whose beauty should, for all time, tell of the joy there was in the making of it; or again, it was the discernment and identification of high-peaks of which the climbing might be, in the years to come, the task of those in whom strenuous labour is a delight and fine air an intoxication." And, truly it is said "this spirit was a new one in England at this time (beginning of the 19th Century) of which we may easily miss the significance."

Dr. Macfarlane has mentioned Sylvester's "greatest mathematical achievement," viz., a strict proof of Newton's hitherto *unproved* Rule, giving an inferior limit to the number of imaginary roots of an equation of *any* degree. The problem had defied the efforts of mathematicians like Euler, Waring, and Maclaurin and the general belief among mathematicians was that Newton himself was not in possossion of other than empirical evidence in support of the Rule given by him in the first chapter on equations in his Arithmetica universalis.

As to the genesis of his researches on Newton's Rule, Sylvester thanked De Morgan for bringing in a marked manner to his notice the original question from which all the rest proceeded. And characteristically added "As all roads are said to lead to Rome, so I find, in my own case at least, that all Algebraical inquiries sooner or later end at the Capitol of Modern Algebra over whose shining portal is inscribed "Theory of Invariants." Sylvester was not only the architect of the Capitol but was also its High priest throughout his long and glorious life!!

As regards the *method* of his discovery Sylvester's statement was characteristic. "I owed my success, he says, chiefly to merging the Theorem to be proved in one of greater scope for generality. In mathematical research, reversing the axiom of Euclid and

controverting the proposition of Hesiod, it is a continual matter of experience that the *whole is less than its part*.

Dr. Macfarlane has given a satisfactory account of the notable work done by Sylvester across the Atlantic, at Baltimore, in founding the American Journal of Mathematics and in "lighting a fire of intellectual interests"—the true object of a University—in the newly-founded John Hopkins University. A highly interesting point in, the lecture is Dr. Macfarlane's quotation of the fine delineation by Dr. F. Franklin, one of Sylvester's pupils at Baltimore, of the inspiring character of Sylvester's teaching. Sylvester often wrote Mathematics in the language of poetry and Dr. Macfarlane has not failed to give notable instances of this characteristic of the poet-mathematician. He has quoted Sylvester's well-known verse "to a missing member of a family group of terms in an algebraic formula" in connection with the development of the "grub" into the "chrysalis" and the "Imago" in his Inaugural Leccture on Reciprocants delivered on his succeeding Stephen Smith as the Savilian Professor at Oxford.

One cannot rise from a perusal of Dr. Macfarlane's lecture without agreeing with M. Nöether when he says that "The exponents of Sylvester's essential characteristics are an *intuitive talent* and a faculty of *invention* to which we owe a series of ideas of *lasting* value and bearing the germs of *fruitful* methods. To no one, more fittingly, than to Sylvester, can be applied one of the mottoes of the Philosophic Magazine :—

Admiratio generat Quaestionem, quaestio investigationem, investigatio inventionem."

And with Dr. H. F. Baker, the editor of Sylvester's collected mathematical works, we may well conclude by saying "Sylvester was, before all an *abstract* thinker, his admiration was ever for intellectual triumphs, his constant worship was of the things of the mind. This it was which seems to have most impressed those who knew him personally. And because of this, his work will endure, according to its value—mingling with the stream fed by the toil of innumerable men—of which the issue is as the source. He is of those to whom it is given to renew in us the sanity which is called faith!"

In his lecture on Kirkman, Dr. Macfarlane has done well in bringing to notice the life and labours of one of the "most penetrating

mathematicians" of the 19th Century, who has not hitherto been very much in the lime-light. One remarkable fact which Dr. Macfaclane has pointed out is that Kirkman was one of the greatest of the British contributors to the Theory of Groups. The archives of the Literary and Philosophical Society of Manchester contain many a contribution from Kirkman and students of the all-unifying Theory of Groups would do well, for the reputation of England, to resuscitate Kirkman's notable memoirs on the complete theory of groups and the Theory of Polyedra.

Like Rowan Hamilton whose intimate friend Kirkman was, he had a metaphysical side to his character. And Dr. Macfarlane has enlivened his lecture by mentioning Kirkman's trenchant criticism of Spencer's Philosophy. Spencer had defined Evolution as "a change from an indefinite in-coherent homogeneity, to a definite coherent heterogeneity, through continuous differentiations and integrations." Kirkman's paraphrase of this was "Evolution is a change from a nohowish untalk aboutable all-likeness, to a somehowish—and—in-general-talk-aboutable not-all-likeness, by continuous somethingelse-i-fications and stick-togetherations"——a paraphrase which though distinctly "polemical and full of sarcasm"—commended itself to such great minds as Tait and Clerk-Maxwell!

Dr. Macfarlane's lecture on Isaac Todhunter fills a gap which was acutely felt by students of Mathematics, who in their early days had been solely nurtured on Todhunter's elementary writings. How many of us, Indians of the last generation, owe to Todhunter our early initiation into the mysteries of Mathematics!

Todhunter's was a singular personality and Dr. Macfarlane deserved our best thanks for presenting before us the fine picture of a Philosopher and a Literateur—a devotee to things of the intellect. Todhunter's fame as a Mathematician, however, rests on a more solid foundation than his early text books and his literary writings. That foundation consists of his unrivalled English works on the history of some branches of mathematics—monuments of careful research and deep learning. Dr. Macfarlane fitly closes his notes of Todhunter by summing up his character in the words of Professor Mayor:—

"Todhunter had no enemies, for he neither coined nor circulated scandal; men of all sects and parties were at home with him, for

he was many-sided enough to see good in every thing. His friendship extended even to the lower creatures. The canaries always hung in his room, for he never forgot to see to their wants." What a fine character Todhunter's was!

When we look back upon the work done by the ten great men noticed above, our outlook of life is widened, faith in human nature is renewed and belief in Divine guidance is confirmed. And looking into the future, in the light gained in an era of unparalleled progress, we cannot but share the optimism of Prof. James Pierpont when he says "an equally bright prospect greets our eyes; on all sides fruitful fields of research invite our labour and promise easy and rich returns."

A. C. BOSE.

On an application of the theory of Functions to Dynamics.

By

Hariprasanna Banerjee.

1. It is generally assumed that, if the force acting on a particle and the initial conditions, are uniquely defined, the position of the particle at any time is uniquely determinate. That this principle is not *always* true, was first pointed out by Poisson* in 1806, by an example. About 20 years ago, many examples illustrating the failure of this principle were given by Professor Painléve.†

The object of my paper is to apply a recent discovery of Dr. O. Perron,‡ in the theory of differential equations, to generalise Poisson's example in a direction utterly different from that in which it was generalised by Professor Painléve. I have shown that, for the same initial conditions and for the same single-valued, finite and continuous force, (1) there may be two possible positions of the particle at a particular time and (2) the particle may take a particular position an infinite number of times, although the motion is *not periodic.*

I should like to express my indebtedness to Dr. Ganesh Prasad at whose suggestion I took up, and under whose guidance I carried on the investigation, the results of which are embodied in this paper.

2. Let us consider a unit particle, moving on the axis of x, under the single-valued and continuous force $f(x)$, where

$$f(x) \text{ is } \frac{1}{2}\frac{d}{dx}\left(x^{\frac{5}{3}}\sin^{\frac{6}{5}}\frac{1}{\sqrt{x}}\right),$$

$$i.e.\ \frac{5}{6}x^{\frac{2}{3}}\sin^{\frac{6}{5}}\frac{1}{\sqrt{x}} - \frac{3}{10}x^{\frac{1}{6}}\sin^{\frac{1}{5}}\frac{1}{\sqrt{x}}\cos\frac{1}{\sqrt{x}},$$

or o, according as x is different from, or equal to, o.

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* See *Journal de l'école polytechnique*, cah. 13, pp. 63 and 106.

† See his "Leçons sur la théorie analytique des équations différentielles," Paris, 1897, p. 549.

‡ Über Differentialgleichungen erster Ordnung, die nicht nach der Ableitung aufgelöst sind" (*Jahresbericht d. d. Math.—Vereinigung, 1918*).

Then, if initially $\frac{dx}{dt} = o$, integrating the equation of motion

$$\frac{d^2x}{dt^2} = f(x),$$

we have

$$\left(\frac{dx}{dt}\right)^2 = x^{\frac{5}{3}} \sin^{\frac{6}{5}} \frac{1}{\sqrt{x}} \qquad (1)$$

$$= \left(x^{\frac{5}{6}} \sin^{\frac{3}{5}} \frac{1}{\sqrt{x}}\right)^2.$$

Further, if initially x is zero, then integrating (1) and keeping in mind the principle enunciated by Dr. Perron, we have an infinite number of integrals of (1), given by

$$t = \pm \int_0^x \frac{dx_1}{x_1^{\frac{5}{6}} \sin^{\frac{3}{5}} \frac{1}{\sqrt{x_1}}} \qquad (2)$$

$$= \pm\, \phi_0(x) \pm \phi_1(x) \pm \phi_2(x) \pm \;\dots\; \pm \phi_n(x) \pm \,.$$

with any arbitrary choice of signs, where

$$\phi_0(x) = \int_{\frac{1}{(g_x+1)^2 \pi^2}}^{x} \frac{dx_1}{x_1^{\frac{5}{6}} \sin^{\frac{3}{5}} \frac{1}{\sqrt{x_1}}},$$

$$\phi_n(x) = \int_{\frac{1}{(g_x+n+1)^2 \pi^2}}^{\frac{1}{(g_x+n)^2 \pi^2}} \frac{dx_1}{x_1^{\frac{5}{6}} \sin^{\frac{3}{5}} \frac{1}{\sqrt{x_1}}},$$

and g_x is the greatest integer in $\frac{1}{\pi\sqrt{x}}$.

3. Before considering the solution given by equation (2), let us note at the outset that a quite different solution of the problem is that given by $x = o$, for *every* value of t.

4. We now consider (2). The motion corresponding to this is non-periodic. For, let

$$t_1, t_2, t_3\ t_4, \ldots \qquad \ldots\ldots \qquad \ldots \qquad \ldots \qquad (3)$$

be the values of t for a particular value of x, say a. Then if the motion were periodic, the difference between any two of the sequence (3), say $t_p - t_q$, must be numerically never less than a finite quantity τ $(>o)$; but we see that this is not true. For, if we take t_p to be the value when all the signs are taken so as to be positive and t_q the value when all the signs except the $(n+1)$th one are positive, then

$$(t_p - t_q) = 2\int_{\frac{1}{(g_a+n+1)^2\pi^2}}^{\frac{1}{(g_a+n)^2\pi^2}} \frac{d\,x_1}{x_1^{\frac{5}{6}} \sin^{\frac{3}{5}} \frac{1}{\sqrt{x_1}}}$$

$$< \frac{C}{(g_a+n)^{\frac{5}{8}}},$$

g_a being the greatest integer in $\dfrac{1}{\pi\sqrt{a}}$ and C, a finite constant independent of n.

As we now make n bigger and bigger $|\ t_p - t_q\ |$ tends to become smaller and smaller, and this difference may be made smaller than τ.

On the vibrations of a membrane whose boundary is an oblique parallelogram.

BY

SASINDRACHANDRA DHAR.

The object of this paper is to investigate the vibrations of a membrane bounded by an *oblique* parallelogram. The method used by me is based on the theory of infinite determinants as developed in recent times by the late Professor H. Poincaré*, Professor Helge Von Koch** and Professor Cazzaniga†.

All the results obtained in this paper are believed to be new, no previous writer having met with any success in solving the problem.

I should like to express my indebtedness to Dr. Ganesh Prasad at whose suggestion I took up, and under whom I carried on, the investigation the results of which are embodied in this paper.

1. Let the sides AB and AD of the parallelogram ABCD be taken as the axes of x and y respectively and let us suppose that the sides AB and AD are of length a and b respectively, the included angle being α.

Referred to rectangular co-ordinates X and Y, the axes of X and Y being AB and AY, the differential equation for the vibration of the membrane is

$$\frac{\delta^2 w}{\delta t^2}=k^2\left(\frac{\delta^2 w}{\delta X^2}+\frac{\delta^2 w}{\delta Y^2}\right). \tag{1}$$

Transforming into oblique axes, we get

$$\sin^2\alpha\ \frac{\delta^2 w}{\delta t^2}=k^2\left(\frac{\delta^2 w}{\delta x^2}-2\cos\alpha\ \frac{\delta^2 w}{\delta x\delta y}+\frac{\delta^2 w}{\delta y^2}\right). \tag{2}$$

The problem will be solved, if we can find a function w, which, being a solution of the above differential equation, vanishes at the same time on the boundary, *i,e.*

$$w=o,\ \text{when } x=o,\ x=a,\ y=o,\ \text{or } y=b.$$

* "Sur les déterminants d'ordre infini" (*Bulletin de la Société mathématique de France*, T. 14.)

** "Sur une application des déterminants infinis à la théorie des équations differentielles linéaires" (*Acta Mathematica* Vols. 15 & 16); also "Sur la convergence des déterminants infinis" (*Rendiconti del Circolo Matematico di Palermo*, Vol. 28).

† "Sui determinanti d'ordine infinito": (*Annali di Matematica*, t. 26.)

Thus if we assume $w = V \cos pt$, V being a function of x and y, the problem reduces to the following :—

To find such a solution V of the differential equation

$$\frac{\delta^2 V}{\delta x^2} - 2 \cos a \frac{\delta^2 V}{\delta x \delta y} + \frac{\delta^2 V}{\delta y^2} + \theta V = 0. \qquad (3)$$

θ being equal to $\frac{p^2}{k^2} \sin^2 a$,

that $V = o$ when $x = o$, $x = a$, $y = o$, or $y = b$.

2. Let us assume V to be of the form :—

$$V = \sum_1^\infty \sum_1^\infty A_{m,n} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y. \qquad (4)$$

The boundary conditions are satisfied by the above expression and in order that it may be a solution of (3), we must have :—

$$\sum_1^\infty \sum_1^\infty \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} - \theta \right) A_{m,n} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{a} y$$

$$+ 2 \cos a \sum_1^\infty \sum_1^\infty \frac{mn\pi^2}{a\,b} A_{m,n} \cos \frac{m\pi}{a} x \cos \frac{n\pi}{b} y = 0 \qquad (5)$$

for all values of x and y lying in the region

$$o \leq x \leq a \text{ and } o \leq y \leq b.$$

Multiply both the sides of (5) by $\sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$ and integrate throughout the area of the membrane. Then we easily obtain

$$c\lambda \sum_{m_1 \neq m} \sum_{n_1 \neq n} R_{m,m_1} R_{n,n_1} A_{m_1,n_1}, + \left[m, n \right] A_{m,n} = 0, \quad (6)$$

where the summation is to be taken for all values of m_1 and n_1 except $m_1 = m$ and $n_1 = n$, and where

$$\left[m, n \right] = \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} - \theta \right), \lambda = \cos a, c = \frac{32}{ab},$$

and $R_{p,q} = \frac{pq}{p^2 - q^2}$ or 0 according as $p+q$ is odd or not.

3. From (6), Art 2, we obtain, by eliminating the A's, an infinite determinant, whence we are to determine the values of θ :—

$$\Delta \equiv \left| \begin{array}{ccccccccc}
[1,1], & c\lambda R_{1,1} R_{1,2}, & c\lambda R_{1,1} R_{1,3}, & \cdots & c\lambda R_{1,2} R_{1,1} & \cdots & c\lambda R_{1,m} R_{1,1}, & c\lambda R_{1,m} R_{1,2}, & \cdots \\
c\lambda R_{1,1} R_{2,1}, & [1,2] \;, & c\lambda R_{1,1} R_{2,3}, & \cdots & c\lambda R_{1,2} R_{2,1} & \cdots & c\lambda R_{1,m} R_{2,1}, & c\lambda R_{1,m} R_{2,2}, & \cdots \\
c\lambda R_{1,1} R_{3,1}, & c\lambda R_{1,1} R_{3,2}, & [1,3] \;, & \cdots & c\lambda R_{1,2} R_{3,1}, & \cdots & c\lambda R_{1,m} R_{3,1}, & c\lambda R_{1,m} R_{3,2}, & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
c\lambda R_{2,1} R_{1,1}, & c\lambda R_{2,1} R_{1,2}, & c\lambda R_{2,1} R_{1,3}, & \cdots & [2,1] & \cdots & c\lambda R_{2,m} R_{1,1}, & c\lambda R_{2,m} R_{1,2}, & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
c\lambda R_{m,1} R_{1,1}, & c\lambda R_{m,1} R_{1,2}, & c\lambda R_{m,1} R_{1,3}, & \cdots & \cdots & \cdots & [m,1] \;, & c\lambda R_{m,m} R_{1,2}, & \cdots \\
c\lambda R_{m,1} R_{2,1}, & c\lambda R_{m,1} R_{2,2}, & c\lambda R_{m,1} R_{2,3}, & \cdots & \cdots & \cdots & \cdots & [m,2] \;, & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array} \right| = 0 \qquad (7)$$

4. The infinite determinant, which we have obtained, must be convergent from physical considerations. It is, however, best to have here a rigorous demonstration.

* It is known that if Δ be an infinite determinant whose diagonal terms are unity, then in order that the infinite determinant may converge, it is sufficient that the series formed with all the elements of the determinant with the exception of the diagonal terms, be absolutely convergent.

Thus the determinant (7) is convergent, if the series

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{n_1 \neq n}^{\infty} \sum_{m_1 \neq m}^{\infty} \frac{S_{m,m_1} S_{n,n_1}}{\left(\frac{\pi^2}{a^2 n^2} + \frac{\pi^2}{b^2 m^2} - \frac{\theta}{m^2 n^2}\right)} \right]$$

be absolutely convergent, $S_{p,q}$ being $\frac{1}{p^2 - q^2}$.

Now since $\sum_{m_1 \neq m}^{\infty} \frac{1}{m^2 - m_1^2} = \frac{-3}{4m^2}$, the series becomes

$$(\tfrac{3}{4})^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\left(\frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} - \theta\right)}.$$

and this will evidently tend towards a limit for all values of θ, except those which are the roots of the equation

$$\prod_{n=1}^{\infty} \prod_{m=1}^{\infty} [m, n] = 0$$

Thus it is proved that the infinite determinant is convergent and that, consequently, it may be treated, for many purposes, as an ordinary determinant.

5. Let us proceed to investigate the behaviour of the A's in the assumed normal function. Suppose that $\theta_{r,s}$ is a root of the determinantal equation (7) and that it is such that it differs from $\left(\frac{r^2 \pi^2}{a^2} + \frac{s^2 \pi^2}{b^2}\right)$ by a very small quantity. Then corresponding to this value of θ, we will get a series of equations like those of (6) to determine the A's.

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* Poincaré and Helge von Koch, *loc. cit.*

Hence, retaining only the first power of λ and considering only the absolute values of the A's, we get the following series of equations to determine the A's:—

$$\frac{A_{1,1}\,[1,1]_{r,s}}{(R_{r,1}\ R_{s,1})} = \frac{A_{1,2}\,[1,2]_{r,s}}{(R_{r,1}\ R_{s,2})} = \ldots = \frac{A_{1,n}\,[1,n]_{r,s}}{(R_{r,1}\ R_{s,n})} =$$

$$\ldots \frac{c\,\lambda\,A_{r,s}}{1} = \ldots \frac{A_{m,n}\,[m,n]_{r,s}}{(R_{r,m}\ R_{s,n})} = \ldots \qquad (8)$$

where $[m,n]_{r,s} = \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} - \theta_{r,s}\right)$.

The series of coefficients *viz.*, $\sum\limits_{n=1}^{\infty} A_{m,n}$ is thus comparable with the series

$$\sum_{n=1}^{\infty} \frac{n}{s^2-n^2}\frac{1}{\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} - \theta_{r,s}\right)}$$ which, in turn, being

comparable with the series $\sum\limits_{1}^{\infty} \frac{1}{n^3}$, is evidently convergent. Similarly

the series $\sum\limits_{m=1}^{\infty} A_{m,n}$ is convergent.

Thus, after a certain finite number of terms, $A_{m,n}$ decreases in absolute value as m and n increase.

It will be seen from (8) that with the exception of the term $A_{r,s}$ which may be chosen arbitrarily, all the terms of the series $\sum\limits_{n=1}^{\infty} A_{r,n}$

and $\sum\limits_{m=1}^{\infty} A_{m,s}$ are zero.

Thus we see that for a particular value $\theta_{r,s}$ of θ, the normal function of the differential equation is

$$V_{r,s} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{A_{m,n}}{A_{r,s}} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \qquad \ldots \qquad (9)$$

the A's being obtained from (6) for the particular value $\theta_{r,s}$ of θ, the co-efficient $A_{r,s}$ being the most important one.

6. From the preceding articles, we draw the following conclusions:—

(i) The normal functions form an orthogonal system. This can be easily shown by making use of Green's theorem and noting that the normal functions vanish on the boundary. Hence we get

$$\int_{o}^{b} \int_{o}^{a} \mathrm{V}m_1,n_1 \, \mathrm{V}m_2,n_2 \, dx \, dy = o, \tag{9}$$

$\mathrm{V}m_1,n_1$ and $\mathrm{V}m_2,n_2$ being two different normal functions.

(ii) From the determinantal equation (7), we deduce that the roots are all real.* It can be easily shown that for small values of λ, the roots are all positive and distinct and we get a doubly infinite system of values of θ.

(iii) It is easily deducible from the determinantal equation that $\frac{\delta p}{\delta a}$ and $\frac{\delta p}{\delta b}$ are always negative. Hence we conclude that the pitch increases as one or both the sides are diminished, a result which is in agreement with the well-known theorem of Schwarz**, viz. that the characteristic frequencies of an area T are less than those of an area T′, if T′ is contained in T.

(iv) It is also found from the determinantal equation that, under certain circumstances, $\frac{\delta p}{\delta a}$ is negative for small values of λ.

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* Burnside and Panton: (*Theory of Equations*, Vol. II, p. 65.)

** *Gesammelte Abhandlungen*, Bd. 1, p. 260.